Deduction and Reduction Theorems for Inferential Erotetic Logic
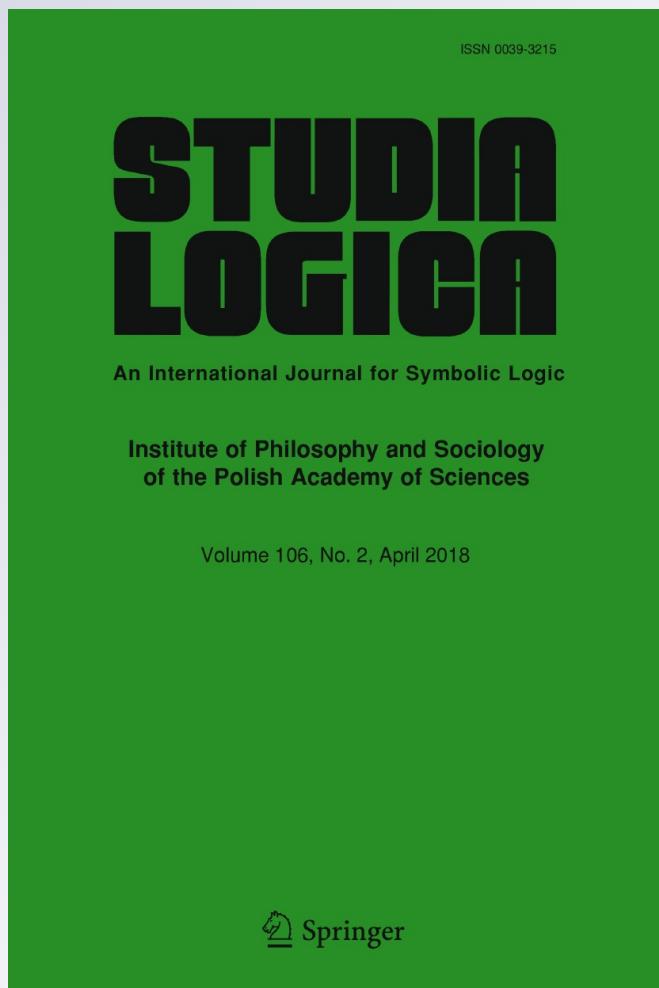
Andrzej Wiśniewski

Springer

Andrzej Wiśniewski

Deduction and Reduction Theorems for Inferential Erotetic Logic

Abstract. The concepts of question evocation and erotetic implication play central role in Inferential Erotetic Logic. In this paper, deduction theorems for question evocation and erotetic implication are proven. Moreover, it is shown how question evocation by a finite non-empty set of declaratives can be reduced to question evocation by the empty set, and how erotetic implication based on a finite non-empty set of declaratives can be reduced to a relation between questions only.

Keywords: Logic of questions, Multiple-conclusion logic, Deduction.

1. Introduction

The logic of questions, sometimes called *erotetic logic*,[1] is a branch of philosophical logic. Many prominent logicians, for example Nuel D. Belnap, Jaakko Hintikka or Johan van Benthem—to mention only a few—devoted books or their substantive chapters to the field. (See [1,8–10], and [17], respectively.) The interest in the logic of questions is currently growing, as witnessed, for instance, by [2,12–14], or the special issue of *Synthese* [6] published in 2015. However, no commonly accepted theory has been worked out so far. The main approaches still differ conceptually and, what is more important, in focussing their interests on different aspects of questions and questioning. Comparing these approaches is not an easy task, which has been accomplished only partially (*cf.* [5,14,22]). Given this, it seems quite justified to address problems by means of the conceptual apparatus of a preferred paradigm, leaving apart the ongoing foundational dispute. This is how we are going to proceed here. In this paper we will be working within Inferential Erotetic Logic (IEL for short). IEL focusses its attention on *inferential* aspects of questioning. The idea of IEL originates from the late 1980s. The monograph [18] summarizes results obtained until the early 1990s. The book [19] presents IEL in its current form.

---

[1]After the Greek word *erotema*, meaning 'question'.

1.1.   Erotetic Inferences and IEL: Aims of This Paper

In many cases arriving at questions resembles coming to conclusions: there
are premises involved and some inferential thought processes take place. In
other words, there exist *erotetic inferences*, that is, thought processes in
which one arrives at a question on the basis of some previously accepted
declarative sentence(s) and/or a previously posed question.

IEL offers an account of *validity* of erotetic inferences. It proceeds as
follows. First, some criteria of validity are proposed, separately for erotetic
inferences that involve only declarative premises and for these in which an
interrogative premise occurs (for details see, e.g., [19], Chapter 5). Then two
semantic relations are defined: *evocation* of questions by sets of declarative
formulas, and *erotetic implication* of a question by a question together with
a (possibly empty) set of declarative formulas. Validity of erotetic inferences
of the consecutive kinds is defined in terms of question evocation and erotetic
implication, respectively.

The role played by question evocation and erotetic implication in IEL
resembles that performed by entailment in a logic of statements. Thus the
properties of question evocation and erotetic implication are worth being
studied. As a matter of fact, a lot is known about them so far (*cf.*, e.g.,
[18,19]). Our aim is to make a further step in this direction. First, we prove
some theorems which seem to deserve the label *deduction theorems* for ques-
tion evocation and erotetic implication. Second, we prove some *reduction
theorems*. As for question evocation, we address the following issue. Sup-
pose that a question Q is evoked by a non-empty set of declarative formulas
X. How can one reduce this to evocation of a question (related both to Q
and X) by the empty set? Concerning erotetic implication, the issue is: how
can one reduce erotetic implication based on a non-empty set of declara-
tive formulas to erotetic implication between questions only, that is, to pure
erotetic implication?

2.   The Logical Basis

We will be working here with IEL in its most general setting presented in
[19], based on Minimal Erotetic Semantics (MiES for short).

Generally speaking, MiES enables an introduction of some important se-
mantic notions pertaining to questions regardless of whether—and if so,
how—the semantics of questions has been previously elaborated in detail.
Moreover, MiES relies upon only few assumptions concerning the syntax of

questions considered and, at the level of declaratives, can be conjoined with (semantics of) both Classical Logic and a wide class of non-classical logics.[2]

2.1.   Syntax

Let $\mathcal{L}$ be a formal language, in which at least two categories of well-formed expressions occur: *declarative well-formed formulas* (hereather: d-wffs) and *interrogative formulas* (hereafter: questions).[3] Generally speaking, $\mathcal{L}$ has thus (possibly among others) a "declarative part" and an "erotetic part." We do not assume in advance what formal language (a non-modal propositional language, a modal propositional language, a first-order language, etc.) performs the role of the declarative part of $\mathcal{L}$. We also stay (almost) neutral concerning the form of questions of $\mathcal{L}$.[4] We assume, however, the existence of an assignment of *direct answers* to questions. More precisely, we stipulate that for each question Q of $\mathcal{L}$ there exists an at least two-element set, $\mathbf{d}Q$, of d-wffs of $\mathcal{L}$, called *the set of direct answers to Q*. Formally, $\mathbf{d}$ is a function that assigns a set of d-wffs to a question. Intuitively, $\mathbf{d}Q$ comprises these possible answers to Q which provide neither less no more information than it is requested by Q. Being true *is not* a prerequisite for being a direct answer.

2.2.   Semantics

In order to define question evocation and erotetic implication we need the concept of multiple-conclusion entailment (see, e.g., [15]), or *mc-entailment* for short. Mc-entailment is a semantic relation between sets of d-wffs, where an entailed set is allowed to contain more than one element. Intuitively speaking, a set of d-wffs X mc-entails a set of d-wffs Y just in case the hypothetical truth of all the d-wffs in X warrants the existence of at least one true d-wff in Y.

Mc-entailment in a language depends on the underlying logic of d-wffs of the language and/or their semantics. But IEL in its general form is neutral in

---

[2] MiES combines some ideas present in Belnap's erotetic semantics (*cf.* [1]) with certain insights taken from the book [15] of Shoesmith and Smiley. Of course, it also goes beyond them. For details of MiES see, e.g., [19], chapters 3 and 4, or [20], pp. 291–309.

[3]Capital letters A, B, C, D, with or without subscripts, will be used below as metalanguage variables for d-wffs, while X, Y, Z are metalanguage variables for sets of d-wffs. The letter Q, and the letter Q with a subscript or a superscript are metalanguage variables for questions.

[4]Due to the lack of agreement as to what questions are, there exist many formalisms for questions; see, e.g., [3, 4, 7, 11, 20] for overviews of logical theories of questions.

the controversy as to what "The Logic" of declaratives is. Below we present a construction which outputs the semantic concept of mc-entailment from the proof-theoretic concept of consequence relation determined by a logic. The construction is not applicable to all cases, but its area of applicability is wide.

2.2.1.   Consequence and Mc-entailment.   Let a logic ℓ, where ℓ is either Classical Logic or a non-classical logic, be the logic of the declarative part of $\mathcal{L}$. In what follows we assume that ℓ is arbitrary but fixed. The logic ℓ determines the corresponding consequence relation among d-wffs of $\mathcal{L}$. Let $\vdash_\ell$ stand for the consequence relation determined by ℓ, satisfying, possibly among others, the standard conditions (*cf.* [15]):

(Overlap) *If $A \in X$, then $X \vdash_\ell A$,*

(Dilution) *If $X \vdash_\ell A$ and $X \subseteq Y$, then $Y \vdash_\ell A$,*

(Cut for Sets) *If $X \cup Y \vdash_\ell A$ and $X \vdash_\ell B$ for every $B \in Y$, then $X \vdash_\ell A$.*

We additionally assume that $\vdash_\ell$ is neither universal nor empty.

   We now define the notion of a $\vdash_\ell$-maximally consistent set. Let $\mathcal{D}_\mathcal{L}$ stand for the set of d-wffs of $\mathcal{L}$, and

$$Y^{\vdash_\ell} =_{df} \{A \in \mathcal{D}_\mathcal{L} : Y \vdash_\ell A\}.$$

DEFINITION 1. ($\vdash_\ell$- *maximally consistent set*) A set of d-wffs Y of $\mathcal{L}$ is $\vdash_\ell$-maximally consistent iff $Y^{\vdash_\ell} \neq \mathcal{D}_\mathcal{L}$ and for each d-wff A of the language: either $A \in Y^{\vdash_\ell}$ or $(Y \cup \{A\})^{\vdash_\ell} = \mathcal{D}_\mathcal{L}$.

   In what follows we assume that $\vdash_\ell$ has the following property:

Lb: *For any set of d-wffs X of $\mathcal{L}$ and any d-wff A of $\mathcal{L}$: if $X \nvdash_\ell A$, then there exists a $\vdash_\ell$-maximally consistent set Y of d-wffs of $\mathcal{L}$ such that $X \subseteq Y$ and $Y \nvdash_\ell A$.*

The property Lb can be called *Lindenbaum feature*[5] or simply Lb-*feature*. Remark that Lb is a property of some, but not all consequence relations.

   In order to pass to the semantic level, we first introduce the concept of partition of the set $\mathcal{D}_\mathcal{L}$ of d-wffs of $\mathcal{L}$; the idea comes from [15]. A *partition*

---

[5]We use here the term "Lindenbaum feature", instead of the more common term "Lindenbaum property", for two reasons. First, we do not consider consequence operations, but consequence relations. Second, we would like to put a stress on the fact that the d-wffs that are not consequences of the "initial" set X are not consequences of the relevant maximal consistent extension(s) of X. (Thus Lb is akin to the concept of Lindenbaum property analysed in [23, pp. 92–93], defined in terms of relative maximality, but with respect to consequence operation and sets closed under the operation; *cf.* also [24, pp. 26–28].)

of $\mathcal{D}_\mathcal{L}$ is an ordered pair $\langle \mathsf{T_P}, \mathsf{U_P} \rangle$, where $\mathsf{T_P} \cap \mathsf{U_P} = \emptyset$ and $\mathsf{T_P} \cup \mathsf{U_P} = \mathcal{D}_\mathcal{L}$. Intuitively, a partition divides d-wffs into "true" (i.e. belonging to $\mathsf{T_P}$) and "untrue" (that is, elements of $\mathsf{U_P}$). However, we are not interested in partitions which do it at random, but only in these partitions of $\mathcal{D}_\mathcal{L}$ which, to speak generally, comply with the logic of d-wffs ℓ by making the consequence relation, $\vdash_\ell$, *truth preserving*. We call these partitions *admissible*.

By assumption, $\vdash_\ell$ has the Lb-feature. ℓ-admissible partitions are defined by:

DEFINITION 2. (*ℓ-admissible partition*) A partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of $\mathcal{D}_\mathcal{L}$ is ℓ-admissible iff $\mathsf{T_P} = Y^{\vdash_\ell}$ for some $\vdash_\ell$-maximally consistent set $Y \subset \mathcal{D}_\mathcal{L}$.

We get:

PROPOSITION 1. *Let $\vdash_\ell$ be a consequence relation that has the Lb-feature. $X \vdash_\ell A$ iff for each ℓ-admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of $\mathcal{D}_\mathcal{L}$: if $X \subseteq \mathsf{T_P}$, then $A \in \mathsf{T_P}$.*

PROOF. ($\Rightarrow$) Assume that there exists an ℓ-admissible partition, P, of $\mathcal{D}_\mathcal{L}$ such that $X \subseteq \mathsf{T_P}$ and $A \notin \mathsf{T_P}$. As P is an ℓ-admissible partition, we have $\mathsf{T_P} = Y^{\vdash_\ell}$ for some $\vdash_\ell$-maximally consistent set Y of d-wffs of $\mathcal{L}$. Suppose that $X \vdash_\ell A$. Therefore, by Dilution, $Y^{\vdash_\ell} \vdash_\ell A$ and hence, again by Dilution, $(Y \cup Y^{\vdash_\ell}) \vdash_\ell A$. But $Y \vdash_\ell B$ for every $B \in Y^{\vdash_\ell}$. Therefore, by Cut for Sets, $Y \vdash_\ell A$, that is, $A \in Y^{\vdash_\ell}$. It follows that $A \in \mathsf{T_P}$. A contradiction.
($\Leftarrow$) Assume that $X \nvdash_\ell A$. Since, by assumption, $\vdash_\ell$ has the Lb-feature, there exists an $\vdash_\ell$-maximally consistent set of d-wffs, Y, such that $X \subseteq Y$ and $Y \nvdash_\ell A$. Hence $\langle Y^{\vdash_\ell}, \mathcal{D}_\mathcal{L} \backslash Y^{\vdash_\ell} \rangle$ is an ℓ-admissible partition of $\mathcal{D}_\mathcal{L}$. By Overlap, $Y \subseteq Y^{\vdash_\ell}$. Since $X \subseteq Y$, it follows that $X \subseteq Y^{\vdash_\ell}$. But $A \notin Y^{\vdash_\ell}$. ∎

The *class of admissible partitions* of $\mathcal{D}_\mathcal{L}$ is defined by:

DEFINITION 3. (*The class of admissible partitions*) Let a logic ℓ be the logic of the declarative part of $\mathcal{L}$. The class of admissible partitions of $\mathcal{D}_\mathcal{L}$ is the class of all ℓ-admissible partitions of $\mathcal{D}_\mathcal{L}$.[6]

We have assumed that $\vdash_\ell$ has the Lb-feature. Given this, and due to Proposition 1, the following can be regarded as the semantic counterpart of the consequence relation $\vdash_\ell$.

DEFINITION 4. (*Entailment*) $X \models_\mathcal{L} A$ iff there is no admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of $\mathcal{D}_\mathcal{L}$ such that $X \subseteq \mathsf{T_P}$ and $A \in \mathsf{U_P}$.

---

[6]Admissible partitions of $\mathcal{D}_\mathcal{L}$ will also be called below simply admissible partitions of $\mathcal{L}$.

Clearly, we have:

COROLLARY 1. $X \models_{\mathcal{L}} A$ *iff* $X \vdash_{\ell} A$.

The semantic relation $\Vdash_{\mathcal{L}}$ of *mc-entailment in* $\mathcal{L}$ can now be defined by:

DEFINITION 5. (*Multiple-conclusion entailment*) $X \Vdash_{\mathcal{L}} Y$ iff there is no admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of $\mathcal{D}_{\mathcal{L}}$ such that $X \subseteq \mathsf{T_P}$ and $Y \subseteq \mathsf{U_P}$.[7]

REMARK As it is common in MiES, we defined entailment and mc-entailment by referring to the class of admissible partitions of (the declarative part of) a language. However, the method of defining the class out of a consequence relation presented above has not been used in MiES so far. Instead, some other methods were employed (*cf.* [19], Chapter 3). Admissible partitions constructed in the way presented in this section are basically proof-theoretic entities. There is no accident in that. When one speaks about deduction theorems, he usually has in mind proof-theoretic results. Although IEL in its current form has a semantic bent, we want to stay as close as possible to proof theory.

2.3. Question Evocation and Erotetic Implication

We are now ready to introduce the concepts of question evocation and erotetic implication. There is no room for an extensive presentation (and discussion) of the underlying intuitions; an interested reader is advised to consult, e.g., [18], Chapter 1, or [19], Chapters 6 and 7.

DEFINITION 6. (*Question evocation*) $\mathbf{E}_{\mathcal{L}}(X, Q)$ iff

1. $X \Vdash_{\mathcal{L}} \mathbf{d}Q$, and

2. for each $A \in \mathbf{d}Q : X \not\Vdash_{\mathcal{L}} \{A\}$.

A question is *sound* iff at least one direct answer to the question is true. Generally speaking, the first clause of Definition 6 amounts to *transmission of truth into soundness*: if only X consists of truths, the question Q must be sound. The second clause amounts to the claim that no single direct answer to Q is entailed by X.

DEFINITION 7. (*Erotetic implication*) $\mathbf{Im}_{\mathcal{L}}(Q, X, Q_1)$ iff

1. for each $A \in \mathbf{d}Q : X \cup \{A\} \Vdash_{\mathcal{L}} \mathbf{d}Q_1$, and

2. for each $B \in \mathbf{d}Q_1$ there exists a non-empty proper subset Y of $\mathbf{d}Q$ such that $X \cup \{B\} \Vdash_{\mathcal{L}} Y$.

---

[7]Observe that $X \models_{\mathcal{L}} A$ iff $X \Vdash_{\mathcal{L}} \{A\}$.

The first clause of Definition 7 warrants the transmission of soundness and truth into soundness. The intuition that underlies the second clause is: each direct answer to an implied question narrows down, together with the respective set X, the class of "possibilities" offered by the whole set of direct answers to the implying question.

Validity of erotetic inferences which do not involve declarative premises can be defined in terms of *pure erotetic implication* (pure e-implication for short). Formally, pure e-implication is erotetic implication based on the empty set of d-wffs.

DEFINITION 8. (*Pure e-implication*) $\mathbf{Im}_{\mathcal{L}}^{\odot}(Q, Q_1)$ iff $\mathbf{Im}_{\mathcal{L}}(Q, \emptyset, Q_1)$.

2.4. The Method

We aim at generality here, so neither syntax nor semantics of a language considered will be described below in detail. In the consecutive sections we will be listing only these specific semantic and syntactic assumptions concerning language $\mathcal{L}$ which enable us to prove the consecutive lemmas and theorems. In other words, we will be indicating *minimal* specific assumptions on which the lemmas and theorems rely upon.

3. The Case of Question Evocation

Minimal assumptions In this section, as well as in Section 4, we assume that the vocabulary of language $\mathcal{L}$ includes the implication connective, $\rightarrow$. We also assume that the following conditions are met:

$\mathsf{CD}_{\rightarrow}$: For each admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of the language:

'$A \rightarrow B$' $\in \mathsf{T_P}$ iff $A \notin \mathsf{T_P}$ or $B \in \mathsf{T_P}$.

$\mathsf{CQ}_{1\rightarrow}$: If Q is a question and C is a d-wff, then there exists a question Q^* such that $\mathbf{d}Q^* = \{C \rightarrow A : A \in \mathbf{d}Q\}$.

$\mathsf{CQ}_{2\rightarrow}$: If Q^* is a question such that for some d-wff C and some set of d-wffs Z: $\mathbf{d}Q^* = \{C \rightarrow B : B \in Z\}$, then there exists a question Q such that $\mathbf{d}Q = Z$.

It is left open what other syntactic and semantic conditions are also met by the language.

Comments Neither condition $\mathsf{CQ}_{1\rightarrow}$ nor condition $\mathsf{CQ}_{2\rightarrow}$ claim that a question can occur as an antecedent or as a consequent of an implication. The

conditions only warrant the existence of the corresponding questions, where the correspondence takes place at the level of the form of direct answers.

Given the generality of the approach presented in Section 2, it cannot be said that condition $\mathsf{CD}_\rightarrow$ always holds. There are languages of the considered kind in which the implication connective does not occur. More importantly, the declarative part of a language can be supplemented with a logic in view of which condition $\mathsf{CD}_\rightarrow$ does not hold. For instance, when we operate with Classical Propositional Logic, the condition holds, but Intuitionistic Propositional Logic provides a counterexample.

The minimal assumptions specified above perform the role of assumptions of lemmas and theorems presented in this section, and in Section 4. For brevity, however, these assumptions will not be explicitly listed each time. Instead, we will be using the label $\mathcal{L}1$. When writing $\models_{\mathcal{L}1}$, we indicate that we are considering mc-entailment in a language fulfilling the above minimal assumptions, and similarly for $\mathbf{E}_{\mathcal{L}1}$, $\mathbf{Im}_{\mathcal{L}1}$, and $\mathbf{Im}^{\odot}_{\mathcal{L}1}$. By admissible partitions we will mean admissible partitions of the language just considered.

3.1.   A Deduction Theorem for Question Evocation

For conciseness, we introduce the following notational convention:

$$|C \rightarrow Y| =_{df} \{C \rightarrow D : D \in Y\}.$$

Let us stress that an inscription $|C \rightarrow Y|$ is not a metalanguage schema of a d-wff, but refers to a set of d-wffs.

One can prove:

LEMMA 1. (Deduction for mc-entailment) *Let* $Y \neq \emptyset$. *Then* $X\cup\{C\} \models_{\mathcal{L}1} Y$ *iff* $X \models_{\mathcal{L}1} |C \rightarrow Y|$.

PROOF. ($\Rightarrow$) Assume that $X \not\models_{\mathcal{L}1} |C \rightarrow Y|$. It follows that there exists an admissible partition, P, such that $X \subseteq \mathsf{T_P}$ and $|C \rightarrow Y| \subseteq \mathsf{U_P}$. Each element of the set $|C \rightarrow Y|$ is of the form $C \rightarrow A$, where $A \in Y$. Therefore, by condition $\mathsf{CD}_\rightarrow$, $C \in \mathsf{T_P}$ and $Y \subseteq \mathsf{U_P}$. It follows that $X \cup \{C\} \not\models_{\mathcal{L}1} Y$.
($\Leftarrow$) Assume that $X \cup \{C\} \not\models_{\mathcal{L}1} Y$. Thus for some admissible partition, P, we have $X \subseteq \mathsf{T_P}$, $C \in \mathsf{T_P}$, and $Y \subseteq \mathsf{U_P}$. Hence, by condition $\mathsf{CD}_\rightarrow$, '$C \rightarrow A$' $\in \mathsf{U_P}$ for every $A \in Y$. It follows that $X \not\models_{\mathcal{L}1} |C \rightarrow Y|$.   ∎

Note that Lemma 1 yields a deduction theorem for entailment: $X\cup\{C\} \models_{\mathcal{L}1} A$ iff $X \models_{\mathcal{L}1} C \rightarrow A$. Moreover, we get:

THEOREM 1. (Deduction for question evocation) $\mathbf{E}_{\mathcal{L}1}(X \cup \{C\}, Q)$ *iff* $\mathbf{E}_{\mathcal{L}1}(X, Q^*)$, *where* $\mathbf{d}Q^* = |C \rightarrow \mathbf{d}Q|$.

PROOF. By Lemma 1. Recall that, by assumption, $\mathbf{d}Q \neq \emptyset$. As for the implication from left to right, the existence of question Q^* is warranted by condition $\mathsf{CQ}_{1\rightarrow}$. In the case of the implication from right to left, question Q exists due to condition $\mathsf{CQ}_{2\rightarrow}$. ∎

According to Theorem 1, a question Q is evoked by a set of d-wffs, Z, that includes a d-wff C if, and only if a "conditionalized" (with regard to C) counterpart of the question Q is evoked by the set $Z\backslash\{C\}$.

3.2. A Reduction Theorem for Question Evocation

For brevity, let us introduce the following notational convention:

$$|\langle C_{i_1},\ldots,C_{i_n}\rangle \rightarrow Y| =_{df} \{C_{i_1} \rightarrow (C_{i_2} \rightarrow (\ldots \rightarrow (C_{i_n} \rightarrow D)\ldots)) : D \in Y\}.$$

By applying Theorem 1 n times we get:

THEOREM 2. (Reduction for question evocation) *Let* $X = \{C_1,\ldots,C_n\}$, *where* $n \geq 1$. *Let* $\langle C_{i_1},\ldots,C_{i_n}\rangle$ *be a sequence without repetitions of all the elements of* X. $\mathbf{E}_{\mathcal{L}1}(X,Q)$ *iff* $\mathbf{E}_{\mathcal{L}1}(\emptyset,Q')$, *where* $\mathbf{d}Q' = |\langle C_{i_1},\ldots,C_{i_n}\rangle \rightarrow \mathbf{d}Q|$.

Thus question evocation by a finite non-empty set of d-wffs reduces, in a sense, to question evocation by the empty set. The price for the reduction is a switch from the "originally" evoked question to its conditionalized counterpart of the above form.

4. The Case of Erotetic Implication

Let us now turn to erotetic implication. The general assumptions we rely on in this section are these specified at the beginning of Section 3 above.

In order to prove a deduction theorem for erotetic implication we need:

LEMMA 2. *Let* $Y \neq \emptyset$. *Then* $X\cup\{D\} \mathrel{\Vert\!\!\models}_{\mathcal{L}1} |C \rightarrow Y|$ *iff* $X\cup\{C \rightarrow D\} \mathrel{\Vert\!\!\models}_{\mathcal{L}1} |C \rightarrow Y|$.

PROOF. ($\Rightarrow$) Assume that $X \cup \{C \rightarrow D\} \mathrel{\Vert\!\!\nvDash}_{\mathcal{L}1} |C \rightarrow Y|$. Thus there exists an admissible partition, P, such that $X \subseteq \mathsf{T}_\mathsf{P}$, '$C \rightarrow D$' $\in \mathsf{T}_\mathsf{P}$, $C \in \mathsf{T}_\mathsf{P}$, and $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$. Hence, by condition $\mathsf{CD}_\rightarrow$, $D \in \mathsf{T}_\mathsf{P}$ and therefore $X \cup \{D\} \mathrel{\Vert\!\!\nvDash}_{\mathcal{L}1} |C \rightarrow Y|$.
($\Leftarrow$) Assume that $X \cup \{D\} \mathrel{\Vert\!\!\nvDash}_{\mathcal{L}1} |C \rightarrow Y|$. Thus for some admissible partition P we have: $X \subseteq \mathsf{T}_\mathsf{P}$, $D \in \mathsf{T}_\mathsf{P}$, and $|C \rightarrow Y| \cap \mathsf{T}_\mathsf{P} = \emptyset$. By condition $\mathsf{CD}_\rightarrow$ we get '$C \rightarrow D$' $\in \mathsf{T}_\mathsf{P}$. Hence $X \cup \{C \rightarrow D\} \mathrel{\Vert\!\!\nvDash}_{\mathcal{L}1} |C \rightarrow Y|$. ∎

LEMMA 3. *Let $Y \neq \emptyset$. Then $(X \cup \{C\}) \cup \{D\} \mathrel{|\!\!\models_{\mathcal{L}1}} Y$ iff $X \cup \{C \to D\} \mathrel{|\!\!\models_{\mathcal{L}1}}$ $|C \to Y|$.*

PROOF. By Lemma 2 and the fact that, due to Lemma 1, $(X \cup \{C\}) \cup$ $\{D\} \mathrel{|\!\!\models_{\mathcal{L}1}} Y$ holds iff $X \cup \{D\} \mathrel{|\!\!\models_{\mathcal{L}1}} |C \to Y|$ is the case.   ■

4.1.   A Deduction Theorem for Erotetic Implication

The following holds:

THEOREM 3. (Deduction for erotetic implication) $\mathbf{Im}_{\mathcal{L}1}(Q, X \cup \{C\}, Q_1)$ *iff* $\mathbf{Im}_{\mathcal{L}1}(Q^*, X, Q_1^*)$, *where* $\mathbf{d}Q^* = |C \to \mathbf{d}Q|$ *and* $\mathbf{d}Q_1^* = |C \to \mathbf{d}Q_1|$.

PROOF. $(\Rightarrow)$ The existence of questions Q^* and Q_1^* is warranted by condition $\mathsf{CQ}_{1\to}$. Recall that $\mathbf{d}Q \neq \emptyset$ and $\mathbf{d}Q_1 \neq \emptyset$.

Assume that $\mathbf{Im}_{\mathcal{L}1}(Q, X \cup \{C\}, Q_1)$. By clause (1) of Definition 7 and Lemma 3, $X \cup \{C \to A\} \mathrel{|\!\!\models_{\mathcal{L}1}} |C \to \mathbf{d}Q_1|$ holds for each $A \in \mathbf{d}Q$. On the other hand, $\mathbf{d}Q_1^* = |C \to \mathbf{d}Q_1|$. Moreover, $\mathbf{d}Q^*$ comprises d-wffs of the form $C \to A$, where $A \in \mathbf{d}Q$.

Let B be an element of $\mathbf{d}Q_1$, and Y be a non-empty proper subset of $\mathbf{d}Q$ such that $(X \cup \{C\}) \cup \{B\} \mathrel{|\!\!\models_{\mathcal{L}1}} Y$. Consider the corresponding element $C \to B$ of $\mathbf{d}Q_1^*$. By Lemma 3, $(X \cup \{C\}) \cup \{B\} \mathrel{|\!\!\models_{\mathcal{L}1}} Y$ iff $X \cup \{C \to B\} \mathrel{|\!\!\models_{\mathcal{L}1}}$ $|C \to Y|$. Hence $X \cup \{C \to B\} \mathrel{|\!\!\models_{\mathcal{L}1}} |C \to Y|$. But $|C \to Y|$ is a non-empty proper subset of $\mathbf{d}Q^*$. Therefore $\mathbf{Im}_{\mathcal{L}1}(Q^*, X, Q_1^*)$.

$(\Leftarrow)$ If $\mathbf{Im}_{\mathcal{L}1}(Q^*, X, Q_1^*)$, then, first, $X \cup \{C \to A\} \mathrel{|\!\!\models_{\mathcal{L}1}} |C \to \mathbf{d}Q_1|$ for any $A \in \mathbf{d}Q$ (note that questions Q and Q_1 exist since condition $\mathsf{CQ}_{2\to}$ holds). Thus, by Lemma 3, $(X \cup \{C\}) \cup \{A\} \mathrel{|\!\!\models_{\mathcal{L}1}} \mathbf{d}Q_1$ for each $A \in \mathbf{d}Q$. Second, $\mathbf{Im}_{\mathcal{L}1}(Q^*, X, Q_1^*)$ yields that for each d-wff of the form $C \to B$, where $B \in$ $\mathbf{d}Q_1$, there exists a non-empty set Y' of d-wffs of the form $C \to A$, where $A \in \mathbf{d}Q$, such that Y' is a proper subset of $\mathbf{d}Q^*$ and $X \cup \{C \to B\} \mathrel{|\!\!\models_{\mathcal{L}1}} Y'$. Observe that Y' equals $|C \to Y|$, where $Y \subseteq \mathbf{d}Q$ and, as Y' is a non-empty proper subset of $\mathbf{d}Q^*$, Y is a non-empty proper subset of $\mathbf{d}Q$. By Lemma 3, $(X \cup \{C\}) \cup \{B\} \mathrel{|\!\!\models_{\mathcal{L}1}} Y$. Yet, B is an arbitrary element of $\mathbf{d}Q_1$. Therefore $\mathbf{Im}_{\mathcal{L}1}(Q, X \cup \{C\}, Q_1)$.   ■

According to Theorem 3, a question Q_1 is implied by a question Q on the basis of a set of d-wffs Z that includes a d-wff C if, and only if, a conditionalized (with regard to C) counterpart of the question Q_1 is implied by the respective conditionalized counterpart of the question Q on the basis of the set of d-wffs $Z \backslash \{C\}$. Remark that both an implied question and an implying question has to be conditionalized in this way.

4.2. A Reduction Theorem for Erotetic Implication

As in the case of question evocation, the deduction theorem yields the corresponding reduction theorem.

THEOREM 4. (Reduction$_1$ for erotetic implication) *Let* $X = \{C_1, \ldots, C_n\}$, *where* $n \geq 1$. *Let* $\langle C_{i_1}, \ldots, C_{i_n} \rangle$ *be a sequence without repetitions of all the elements of* X. *Then* $\mathbf{Im}_{\mathcal{L}1}(Q, X, Q_1)$ *iff* $\mathbf{Im}_{\mathcal{L}1}^{\odot}(Q^\star, Q_1^\star)$, *where* $\mathbf{d}Q^\star = |\langle C_{i_1}, \ldots, C_{i_n} \rangle \to \mathbf{d}Q|$ *and* $\mathbf{d}Q_1^\star = |\langle C_{i_1}, \ldots, C_{i_n} \rangle \to \mathbf{d}Q_1|$.

PROOF. By Theorem 3 and Definition 8. More precisely, we apply Theorem 3 n times, starting with C_{i_n}. Recall that $\{C_1, \ldots, C_n\} = \{C_{i_1}, \ldots, C_{i_n}\}$. ∎

Thus one can transform erotetic implication based on a finite non-empty set of d-wffs into pure e-implication, yet on the price of conditionalizing the implying question as well as the implied question. Since Theorem 4 has the form of a biconditional, a move in the other direction is also possible.

5. Another Reduction Theorem for Erotetic Implication

Belnap introduces in [1, p. 98] the concepts of "added-condition questions" and "added-conjunct questions." The former have direct answers generated by uniformly adding a condition to a given set of direct answers. An added-conjunct question, in turn, has direct answers which result from direct answers to some other question by adding to them the content of the respective "given-that" clause.[8] More precisely, Q^* is an added-conjunct question iff for some d-wff A, $\mathbf{d}Q^* = \{A \wedge B : B \in \mathbf{d}Q\}$, where Q is a question for which the following condition holds: there is no d-wff D such that each direct answer to Q has the form of a conjunction with D as a conjunct.

The deduction and reduction theorems presented above rely, among others, on condition $\mathsf{CQ}_{1\to}$. What the condition means in view of Belnap's proposal is this: an added-condition question exists if the respective "basic" question exists. Let us now consider a similar yet different condition:

---

[8]As for natural languages, added-condition questions usually manifest as the so-called hypothetical questions, e.g.: "If you were to go, would you take an umbrella?". According to Belnap, the following are direct answers to the above hypothetical question: "If I were to go, I would take an umbrella" and "If I were to go, I would not take an umbrella".

Added-conjunct questions usually manifest as "given-that" questions, e.g.: "Given that you are going, are you taking an umbrella?". The meaning of the affirmative answer to this given-that question is "I am going and I am taking an umbrella", while the negative answer means "I am going, but I am not taking an umbrella".

$\mathsf{CQ_{1\wedge}}$: If C is a d-wff and Q is a question which is not an added-conjunct question, then there exists a question Q^* such that $\mathbf{d}Q^* = \{C \wedge B : B \in \mathbf{d}Q\}$.

Condition $\mathsf{CQ_{1\wedge}}$ warrants the existence of an added-conjunct question based on a question which itself is not an added-conjunct question.

Let us now assume that the conjunction connective, $\wedge$, occurs in $\mathcal{L}$, and that, besides condition $\mathsf{CQ_{1\wedge}}$, the following conditions are met:

$\mathsf{CD_{\wedge}}$: For each admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of the language:
'$A \wedge B$' $\in \mathsf{T_P}$ iff $A \in \mathsf{T_P}$ and $B \in \mathsf{T_P}$.

$\mathsf{CQ_{2\wedge}}$: If Q^* is a question such that for some d-wff C and some set of d-wffs Z: $\mathbf{d}Q^* = \{C \wedge B : B \in Z\}$, then there exists a question Q such that $\mathbf{d}Q = Z$.

Similarly as in the previous sections, we will be using a specific label, namely $\mathcal{L}2$, to indicate that the results presented in this section rely on the above minimal assumptions. We adopt the following notational conventions:

$$|C \wedge Y| =_{df} \{C \wedge D : D \in Y\}.$$
$$|\langle C_{i_1}, \ldots, C_{i_n} \rangle \wedge Y| =_{df} \{C_{i_1} \wedge (C_{i_2} \wedge (\ldots \wedge (C_{i_n} \wedge D)\ldots)) : D \in Y\}.$$

We use $\bigwedge\{C_1, \ldots, C_n\}$ for a conjunction of all the d-wffs in $\{C_1, \ldots, C_n\}$. Needless to say, $\bigwedge\{C\} = C$. One can easily prove:

LEMMA 4. *Let $Y \neq \emptyset$. Let $\langle C_{i_1}, \ldots, C_{i_n} \rangle$ be a sequence without repetitions of all the elements of $\{C_1, \ldots, C_n\}$, where $n \geq 1$. Then $\{C_1, \ldots, C_n\} \cup \{D\} \models_{\mathcal{L}2} Y$ iff $\bigwedge\{C_1, \ldots, C_n\} \wedge D \models_{\mathcal{L}2} |\langle C_{i_1}, \ldots, C_{i_n} \rangle \wedge Y|$.*

The following is true:

THEOREM 5. *(Reduction$_2$ for erotetic implication) Let $X = \{C_1, \ldots, C_n\}$, where $n \geq 1$. Let $\langle C_{i_1}, \ldots, C_{i_n} \rangle$ be a sequence without repetitions of all the elements of X. Assume that neither Q nor Q_1 is an added-conjunct question. Then $\mathbf{Im}_{\mathcal{L}2}(Q, X, Q_1)$ iff $\mathbf{Im}_{\mathcal{L}2}^{\circledcirc}(Q^{\circ}, Q_1^{\circ})$, where $\mathbf{d}Q^{\circ} = |\langle C_{i_1}, \ldots, C_{i_n} \rangle \wedge \mathbf{d}Q|$ and $\mathbf{d}Q_1^{\circ} = |\langle C_{i_1}, \ldots, C_{i_n} \rangle \wedge \mathbf{d}Q_1|$.*

PROOF. By Lemma 4 and Definition 7. Recall that $\mathbf{d}Q \neq \emptyset$ and $\mathbf{d}Q_1 \neq \emptyset$. Observe that Z is a non-empty proper subset of $\mathbf{d}Q$ iff $|\langle C_{i_1}, \ldots, C_{i_n} \rangle \wedge Z|$ is a non-empty proper subset of $\mathbf{d}Q^{\circ}$. If questions Q and Q_1 exist, then, by condition $\mathsf{CQ_{1\wedge}}$, questions Q° and Q_1° exist as well. If questions Q° and Q_1° exist, then Q and Q_1 exist as well due to condition $\mathsf{CQ_{2\wedge}}$. ∎

As previously, it cannot be said that the minimal assumptions specified above hold universally. However, Theorem 5 can be viewed as describing

reduction results for logics in which implication does not fulfil condition $CD_{\rightarrow}$, but, nevertheless, conjunction behaves in the standard manner.

6. Summary and a Remark

Generally speaking, we have shown that, provided that some assumptions are met, question evocation is retained when: (a) removing a d-wff from an evoking set is accompanied by conditionalizing the evoked question by the removed d-wff, that is, forming the corresponding hypothetical question, (b) removing the condition of an evoked hypothetical question is accompanied by enriching the evoking set with the removed condition (*cf.* Theorem 1). Erotetic implication, in turn, is retained when: (a) removing an element from an implying set of d-wffs is accompanied by conditionalizing both the implying question and the implied question by the removed d-wff, (b) assuming that the implied and the implying questions share a condition, removing it is accompanied by enriching the implying set of d-wffs with the condition (*cf.* Theorem 3). Question evocation by a finite non-empty set of d-wffs coincides with evocation of the corresponding 'conditional' question(s) by the empty set (*cf.* Theorem 2), while erotetic implication based on a finite non-empty set of d-wffs is reducible to pure e-implication among the corresponding questions (*cf.* Theorems 4 and 5).

The final remark is this. Recall that the role played in IEL by the concepts of question evocation and erotetic implication resembles that played by entailment in a logic of statements. Given the analogy, it seems worthwhile to build, at the proof-theoretic level, formal systems whose theorems describe what questions are evoked by what sets of d-wffs, and what questions are implied by what questions on the basis of what sets of d-wffs. So far there exist systems of this kind for question evocation (for the classical propositional case only; see [16,21]), but there is no system for erotetic implication. The reduction theorems for erotetic implication presented above open a new perspective: in order to build a system for erotetic implication one may consider first—or even only—the conceptually (though not computationally) simpler case of pure e-implication.

References

[1] BELNAP, N. D., and Th. P. STEEL, *The Logic of Questions and Answers*, Yale University Press, New Haven, 1976.

[2] CIARDELLI, I., *Questions in Logic*, ILLC, Amsterdam, 2015.

[3] GINZBURG, J., Questions: Logic and interactions, in J. van Benthem and A. ter Meulen (eds.), *Handbook of Logic and Language*, 2nd edn, Elsevier, Amsterdam, 2011, pp. 1133–1146.

[4] GROENENDIJK, J., and M. STOKHOF, Questions, in J. van Benthem and A. ter Meulen (eds.), *Handbook of Logic and Language*, 2nd edn, Elsevier, Amsterdam, 2011, pp. 1059–1132.

[5] HAMAMI, Y., *The interrogative model of inquiry meets dynamic epistemic logic.* Master's thesis, ILLC, University of Amsterdam, 2010.

[6] HAMAMI, Y., and F. ROELOFSEN, guest editors. Logics of questions, *Synthese* 192, Issue 6, 2015.

[7] HARRAH, D., The logic of questions, in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, vol. 8, 2nd edn, Kluwer, Dordrecht/Boston/London, 2002, pp. 1–60.

[8] HINTIKKA, J., *Semantics of Questions and the Questions of Semantics*, vol. 28 of *Acta Philosophica Fennica*, North-Holland, Amsterdam, 1976.

[9] HINTIKKA, J., *Inquiry as Inquiry: A Logic of Scientific Discovery*, Kluwer, Dordrecht/Boston/London, 1999.

[10] HINTIKKA, J., *Socratic Epistemology: Explorations of Knowledge-Seeking by Questioning*, Cambridge University Press, Cambridge, 2007.

[11] KRIFKA, M., Questions, in K. von Helsinger, C. Maieborn and P. Portner (eds.), *Semantics. An International Handbook of Natural Language Meaning*, vol. II, Mouton de Gruyter, Berlin, 2011, pp. 1742–1785.

[12] ŁUPKOWSKI, P., *Logic of Questions in the Wild. Inferential Erotetic Logic in Information Seeking Dialogue Modelling*, College Publications, London, 2016.

[13] ONEA, E., *Potential Questions at the Semantics-Pragmatics Interface*, Brill, Leiden/Boston, 2016.

[14] PELIŠ, M., *Inferences with Ignorance: Logics of Questions. Inferential Erotetic Logic & Erotetic Epistemic Logic*, Karolinum Press, Prague, 2016.

[15] SHOESMITH, D. J., and T. J. SMILEY, *Multiple-conclusion Logic*, Cambridge University Press, Cambridge, 1978.

[16] SKURA, T., and A. WIŚNIEWSKI, A system for proper multiple-conclusion entailment, *Logic and Logical Philosophy* 24:241–253, 2015.

[17] VAN BENTHEM, J., *Logical Dynamics of Information and Interaction*, Cambridge University Press, Cambridge, 2011.

[18] WIŚNIEWSKI, A., *The Posing of Questions: Logical Foundations of Erotetic Inferences*, Kluwer, Dordrecht/Boston/London, 1995.

[19] WIŚNIEWSKI, A., *Questions, Inferences, and Scenarios*, College Publications, London, 2013.

[20] WIŚNIEWSKI, A., Semantics of questions, in S. Lappin and Ch. Fox (eds.), *The Handbook of Contemporary Semantic Theory*, 2nd edn, Wiley-Blackwell, Oxford, 2015, pp. 273–313.

[21] WIŚNIEWSKI, A., An axiomatic account of question evocation. The propositional case, *Axioms* 5(2):14, 2016.

[22] WIŚNIEWSKI, A., and D. LESZCZYŃSKA-JASION, Inferential erotetic logic meets inquisitive semantics, *Synthese* 192:1585–1608, 2015.

[23] WÓJCICKI, R., *Lectures on Propositional Calculi*, Ossolineum, Wrocław, 1984.

[24] WÓJCICKI, R., *Theory of Logical Calculi. Basic Theory of Consequence Operations*, Kluwer, Dordrecht/Boston/London, 1988.

A. WIŚNIEWSKI
Department of Logic and Cognitive Science
Institute of Psychology
Adam Mickiewicz University
Poznań
Poland
Andrzej.Wisniewski@amu.edu.pl