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GENERALIZED ENTAILMENTS

Abstract. A semantic relation between a family of sets of formulas and a set of formulas, dubbed generalized entailment, and its subrelation, called constructive generalized entailment, are defined and examined. Entailment construed in the usual way and multiple-conclusion entailment can be viewed as special cases of generalized entailment. The concept of constructive generalized entailment, in turn, enables an explication of some often used notion of interrogative entailment, and coincides with inquisitive entailment at the propositional level. Some interconnections between constructive generalized entailment and Inferential Erotetic Logic are also analysed.

Keywords: entailment; families of sets; logic of questions

1. Basic intuitions

As for logic, entailment is most often conceived of as a relation between a set of well-formed formulas (wffs for short) on the one hand, and a single wff on the other. Entailment ensures *transmission of truth*: a wff A entailed by a set of wffs X must be true if only all the wffs in X are true. What 'must' means here depends on a logic under consideration, and similarly for 'truth.' The transmission of truth principle falls under the general schema:

(1.1) *for each* $\mathfrak{A}$*: if all the wffs in* X *are true in* $\mathfrak{A}$*, then* A *is true in* $\mathfrak{A}$*.*

where $\mathfrak{A}$ stands, depending on a case, for: 'valuation' (of an appropriate kind), 'model', 'intended model', 'world of a model', and so forth.

Entailment understood in the standard way exhibits a kind of asymmetry: what is entailed is a single wff, while what is entailing it is a set of wffs. If, for some reasons, you prefer symmetry over the lack of it,

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there are two possible ways of making entailment a relation between sets of wffs. Let X and Y stand for sets of wffs. One may define entailment of Y from X by imposing either of the following conditions:

(1.2) *for each $\mathfrak{A}$: if all the wffs in X are true in $\mathfrak{A}$, then all the wffs in Y are true in $\mathfrak{A}$,*

(1.3) *for each $\mathfrak{A}$: if all the wffs in X are true in $\mathfrak{A}$, then at least one wff in Y is true in $\mathfrak{A}$.*

The condition (1.2) leads to a trivial generalization. Obviously, the condition is fulfilled if, and only if X entails every wff in Y. But the case of condition (1.3) is different. A generalization by the condition (1.3) gives a well-known concept of *multiple-conclusion entailment*, or mc-entailment for short.[1] One cannot say that mc-entailment is always definable in terms of entailment. The following observations justify this claim. First, it happens that a set of wffs is mc-entailed, but no wff in the set is entailed. This phenomenon shows up even at the elementary level of classical propositional logic. Here is a simple example. The singleton set $\{p \vee q\}$ mc-entails the set $\{p, q\}$, but neither p nor q is entailed by $\{p \vee q\}$. Second, it is not a general rule that mc-entailment of Y from X reduces to entailment of $\bigvee Y$ (that is, a disjunction of all the wffs in Y) from X. It can happen that Y is an infinite set and the corresponding language lacks infinite disjunctions. More importantly, there are non-classical logics in which mc-entailment of Y from X holds, but entailment of $\bigvee Y$ from X does not hold (see [21] for examples).

Mc-entailment, however, exhibits a kind of asymmetry with respect to quantifiers used. As for the condition (1.3), the clause occurring in the scope of 'for each $\mathfrak{A}$' involves universal quantifier in the antecedent and existential quantifier in the consequent. When X mc-entails Y, one expects from X to *consists of* truths, while Y is only required to *contain* a truth. This quantificational heterogeneity shows that X and Y are intuitively understood in different manners. A set of wffs can represent a belief base, but can also represent a search space. It seems natural to think of a mc-entailed set as representing a search space. On the other hand, when X mc-entails Y, it seems natural to construe X as a representative of a (potential) belief base.

[1] One can find this concept already in Gentzen [9] and Carnap [3]. Its full metalogical elaboration is due to Scott [19], and a throughout analysis can be found in the monograph of Shoesmith and Smiley [20].

But what if we are after a relation between sets of wffs each of which represents a search space? At the first step we can consider a relation between sets of wffs, X and Y, fulfilling the following condition:

(1.4) *for each $\mathfrak{A}$: if at least one wff in X is true in $\mathfrak{A}$, then at least one wff in Y is true in $\mathfrak{A}$.*

Generally speaking, condition (1.4) expresses the following intuition: if truth can be found in a search space X, then truth can be found in the search space Y.

A natural generalization would be to allow for many search spaces in the antecedent. Let Φ be a family[2] of sets of wffs, and let Y be a set of wffs. We may require Φ and Y be connected according to the following principle:

(1.5) *for each $\mathfrak{A}$: if, for all $X \in \Phi$, at least one wff in X is true in $\mathfrak{A}$, then at least one wff in Y is true in $\mathfrak{A}$.*

Now the intuition is: if truth can be found in all the search spaces that belong to Φ, then truth can be found in the search space Y as well.

As for condition (1.5), existential quantifier plays the crucial role both in the antecedent and the consequent. But there is a price: we have jumped to the level of families of sets. Moreover, homogeneity of the domain and the range is lost.

1.1. Aims and summary

In this paper we define and investigate two relations between families of sets of wffs and sets of wffs. We dub the first relation *generalized entailment*. It is defined and analysed in Section 3. Its definition matches the intuition expressed by condition (1.5) specified above. As we show, basic properties of generalized entailment are akin to these of "standard" entailment. We also show that entailment and mc-entailment can be viewed as special cases of generalized entailment. Then we prove that, for any family of non-empty sets of wffs, being entailed (in the sense of generalized entailment) by the family amounts to being mc-entailed by each set which contains exactly one representative of every set that belongs to the family. In Section 4 we define and examine some subrelation of generalized entailment, which, as we show, deserves the

[2] By a family of sets we mean, here and below, a set of sets.

name *constructive generalized entailment*. Section 5, generally speaking, is devoted to bridges between constructive generalized entailment, the logic of questions, and inquisitive semantics. In subsection 5.4 we show that some notion of interrogative entailment, occurring quite often in the literature on questions and questioning, can be explicated in terms of constructive generalized entailment. In subsection 5.3 we point out that the concepts of reducibility of questions to sets of questions, elaborated on within Inferential Erotetic Logic, are closely connected with interrogative entailment explicated in the above-mentioned manner. Constructive generalized entailment is also related to inquisitive entailment. In subsection 5.4 we prove that inquisitive entailment based on the (current version of) inquisitive propositional logic InqB and constructive generalized entailment based on classical propositional logic coincide set-theoretically, given that we restrict ourselves to the so-called sets of resolutions of wffs. Subsection 5.5 describes another link between constructive generalized entailment and current research in the logic of questions. We show that the so-called erotetic search scenarios, a formal tool developed within Inferential Erotetic Logic, are useful in establishing which sets of wffs are entailed, in the sense of constructive generalized entailment, by which families of sets of wffs.

2. Logical preliminaries

We consider a formal language for which the concept of well-formed formula (wff) is defined. We use A, B, C, D as metalanguage variables for wffs, and X, Y, W, Z, with subscripts if needed, as metalanguage variables for sets of wffs. The Greek letters Φ, Ψ will refer to families of sets of wffs, that is, sets of sets of wffs. In the metatheory we assume a version of set theory that allows both for sets and classes, and incorporates the Axiom of Choice. We use standard set-theoretic terminology and notation. The expression "iff" abbreviates "if and only if."

As for the general semantic framework, we follow here the idea of [20], yet with some minor adjustments.

Let $\mathcal{D}_{\mathcal{L}}$ be the set of wffs of a formal language $\mathcal{L}$. A *partition* of $\mathcal{D}_{\mathcal{L}}$ is an ordered pair:

$$\mathsf{P} = \langle \mathsf{T}_{\mathsf{P}}, \mathsf{U}_{\mathsf{P}} \rangle,$$

where $\mathsf{T}_{\mathsf{P}} \cap \mathsf{U}_{\mathsf{P}} = \emptyset$ and $\mathsf{T}_{\mathsf{P}} \cup \mathsf{U}_{\mathsf{P}} = \mathcal{D}_{\mathcal{L}}$.

We assume that the language considered is supplemented with semantics rich enough to define some concept of truth for wffs. The concept is always relative to some metalogical constructs, such as valuations, models, matrices, etc. The relevant concept of truth determines the *class of admissible partitions* of the language under consideration. The following examples illustrate this.

EXAMPLE 1. Let $\mathcal{L}$ be the language of classical propositional logic (hereafter: CPL). A *Boolean valuation* is a function v that assigns a truth value, $\mathbf{1}$ or $\mathbf{0}$, to each propositional variable and is extended to all wffs in the standard manner by using the Boolean functions corresponding to the connectives.

A partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$ of $\mathcal{D}_\mathcal{L}$ is *admissible* iff there exists a Boolean valuation v such that $\mathsf{T}_\mathsf{P} = \{A \in \mathcal{D}_\mathcal{L} : v(A) = \mathbf{1}\}$.

Thus, for any admissible partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$, the set T_P comprises all the wffs which are true in the corresponding Boolean valuation v, and (as $\mathsf{U}_\mathsf{P} = \mathcal{D}_\mathcal{L} \setminus \mathsf{T}_\mathsf{P}$), U_P contains the wffs which are false w.r.t. v.

EXAMPLE 2. We consider the propositional modal logic **S4**. Let $\mathcal{D}_\mathcal{L}$ be the set of wffs of the language of (propositional) **S4**. The concept of **S4**-Kripke model, as well as the concept of truth of a wff in a world of a model, are defined in the standard manner. We write $(\mathcal{M}, w) \models A$ for 'A is true in world w of model $\mathcal{M}$.'

A partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$ of $\mathcal{D}_\mathcal{L}$ is *admissible* iff for some **S4**-Kripke model $\mathcal{M} = \langle W, R, V \rangle$ and some $w \in W$: $\mathsf{T}_\mathsf{P} = \{A \in \mathcal{D}_\mathcal{L} : (\mathcal{M}, w) \models A\}$.

EXAMPLE 3. This time we assume that $\mathcal{D}_\mathcal{L}$ is the set of wffs of first-order logic (hereafter: FOL). The concepts of FOL-model and of truth of a wff in a FOL-model are defined in the standard manner. By $Ver(\mathcal{M})$ we designate the set of all wffs which are true in a FOL-model $\mathcal{M}$.

A partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$ of $\mathcal{D}_\mathcal{L}$ *is admissible* iff $\mathsf{T}_\mathsf{P} = Ver(\mathcal{M})$, for some FOL-model $\mathcal{M}$.

Classes of admissible admissible partitions of languages different from these just considered can be defined according to the pattern applied above.

When $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$ is an admissible partition, we may think of T_P as the set of truths of the partition, and of U_P as the set of untruths of the partition.

Given that the class of admissible partitions is fixed, "standard" entailment, $\models$, and mc-entailment, $\Vert\!\models$, can be defined by:

Definition 1 (Entailment). $X \models A$ iff there is no admissible partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$ such that $X \subset \mathsf{T}_\mathsf{P}$ and $A \notin \mathsf{T}_\mathsf{P}$.

Definition 2 (Mc-entailment). $X \Vdash Y$ iff there is no admissible partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$ such that $X \subset \mathsf{T}_\mathsf{P}$ and $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$.

For example, in the case of CPL we get: $X \models A$ iff $v(A) = \mathbf{1}$ for every Boolean valuation v such that $v(B) = \mathbf{1}$ for any $B \in X$. As for mc-entailment, we have: $X \Vdash Y$ iff for each Boolean valuation v in which $v(B) = \mathbf{1}$ for every $B \in X$, there exists $A \in Y$ such that $v(A) = \mathbf{1}$.

In what follows, we assume that the language for which we define generalized entailment and the remaining concepts, is an arbitrary but fixed formal language satisfying the general conditions specified in this section. By admissible partitions we mean admissible partitions of the language.

3. Generalized entailment

Generalized entailment (g-entailment for short) is a relation between a family of sets of wffs on the one hand, and a set of wffs on the other. We use $|||\!\models$ as the symbol for g-entailment.

Definition 3 (G-entailment). $\Phi \,|||\!\models Y$ iff for each admissible partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$ such that:

($\star$) for each $X \in \Phi : X \cap \mathsf{T}_\mathsf{P} \neq \emptyset$

it holds that $Y \cap \mathsf{T}_\mathsf{P} \neq \emptyset$.

The proposed definition of g-entailment expresses, in the current conceptual setting, the idea that lies behind condition (1.5) specified in Section 1 above.

3.1. Some examples

Some examples can be helpful.

Example 4. As for CPL, we have: $\Phi \,|||\!\models Y$ iff there is no Boolean valuation v such that:

- for each $X \in \Phi$, $v(A) = \mathbf{1}$ for some $A \in X$, and $v(B) = \mathbf{0}$ for all $B \in Y$.

For instance, the following holds (p, q, r, s, t, u are, here and below, propositional variables):

$$\{\{p \vee q \to s \vee r, p \vee q \to s \vee t\}, \{p, q\}\} \,|||\!\models \{s, r, t\}.$$

EXAMPLE 5. Consider the case of FOL. We have: $\Phi \,|\!|\!\models Y$ iff for each FOL-model $\mathcal{M}$:

- if $X \cap Ver(\mathcal{M}) \neq \emptyset$ for each $X \in \Phi$, then $Y \cap Ver(\mathcal{M}) \neq \emptyset$.

For instance ($\mathtt{P}, \mathtt{S}$ are one-place predicates, and $\mathtt{a}, \mathtt{b}$ are individual constants):

$$\{\{\exists x \mathtt{P}x, \exists x \mathtt{S}x\}, \{\forall x(\mathtt{P}x \vee \mathtt{S}x \rightarrow x = \mathtt{a} \vee x = \mathtt{b}\}\} \,|\!|\!\models \{\mathtt{Pa}, \mathtt{Pb}, \mathtt{Sa}, \mathtt{Sb}\}.$$

Note that $\{\forall x(\mathtt{P}x \vee \mathtt{S}x \rightarrow x = \mathtt{a} \vee x = \mathtt{b}\}\}$ is a singleton set. However, it is not excluded that Φ contains singleton sets.

3.2. Basic properties of generalized entailment

Recall that $|\!|\!\models$ is a relation between a family of sets of wffs and a set of wffs. Interestingly enough, $|\!|\!\models$ still behaves in a "consequence-like" manner.

PROPOSITION 1. *$\Phi \,|\!|\!\models X$ for every $X \in \Phi$.*

PROOF. Suppose otherwise. Then there exists an admissible partition, P, such that both $X \cap \mathsf{T}_\mathsf{P} = \emptyset$ and $X \cap \mathsf{T}_\mathsf{P} \neq \emptyset$. A contradiction. □

PROPOSITION 2. *If $\Phi \,|\!|\!\models Y$ and $\Phi \subset \Psi$, then $\Psi \,|\!|\!\models Y$.*

PROOF. Suppose that $\Psi \,|\!|\!\not\models Y$. Thus there exists an admissible partition, P, such that $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$ and for each $Z \in \Psi : Z \cap \mathsf{T}_\mathsf{P} \neq \emptyset$. As $\Phi \subseteq \Psi$, it follows that $\Phi \,|\!|\!\not\models Y$. □

PROPOSITION 3. *If $\Phi \,|\!|\!\models Y$ and $\Psi \,|\!|\!\models X$ for every $X \in \Phi$, then $\Psi \,|\!|\!\models Y$.*

PROOF. Suppose that $\Psi \,|\!|\!\not\models Y$. So for some admissible partition, P, we have $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$ and $Z \cap \mathsf{T}_\mathsf{P} \neq \emptyset$ for any $Z \in \Psi$. Since $\Psi \,|\!|\!\models X$ for each $X \in \Phi$, it follows that $X \cap \mathsf{T}_\mathsf{P} \neq \emptyset$ for each $X \in \Phi$. Hence $Y \cap \mathsf{T}_\mathsf{P} \neq \emptyset$, as $\Phi \,|\!|\!\models Y$. A contradiction. □

3.2.1. Special cases and cut

Let us introduce:

DEFINITION 4 (Safeset). Y is a safeset iff $Y \cap \mathsf{T}_\mathsf{P} \neq \emptyset$ for each admissible partition $\mathsf{P} = \langle \mathsf{T}_\mathsf{P}, \mathsf{U}_\mathsf{P} \rangle$.

A set containing a valid[3] wff is a safeset. But there exist safesets which do not contain valid wffs. For instance, $\{p, \neg p\}$ is a safeset in view of CPL.

Clearly, we have:

Corollary 1. *A safeset is g-entailed by any family of sets of wffs.*

Proposition 4. *If* $\emptyset \Vert\!\models Y$*, then* Y *is a safeset.*

Proof. Assume that $\emptyset \Vert\!\models Y$. Let P be an arbitrary but fixed admissible partition. The condition $(\star)$ of Definition 3 is (trivially) true w.r.t. $\Phi = \emptyset$. We have:

$(\star')$ for each $X \in \emptyset$: $X \cap \mathsf{T}_{\mathsf{P}} \neq \emptyset$.

Hence $Y \cap \mathsf{T}_{\mathsf{P}} \neq \emptyset$. But P is an arbitrary admissible partition. □

Thus the empty set g-entails only safesets. But if the empty set is an element of a family of sets of wffs, the family g-entails any set of wffs. This is due to:

Proposition 5. *If* $\emptyset \in \Phi$*, then* $\Phi \Vert\!\models Y$ *for any set of wffs* Y*.*

Proof. Assume that $\Phi \Vert\!\not\models Y$ for some set of wffs Y. Thus there exists an admissible partition, P, for which the following condition holds:

$$\textit{for each } X \in \Phi : X \cap \mathsf{T}_{\mathsf{P}} \neq \emptyset\,, \tag{1}$$

but $Y \cap \mathsf{T}_{\mathsf{P}} = \emptyset$. However, $\emptyset \in \Phi$ and hence the condition (1) yields:

$$\emptyset \cap \mathsf{T}_{\mathsf{P}} \neq \emptyset$$

which is impossible. □

G-entailment has a property akin to cut:

Proposition 6. *If* $\Psi \Vert\!\models X$ *and* $\Phi \cup \{X\} \Vert\!\models Y$*, then* $\Psi \cup \Phi \Vert\!\models Y$*.*

Proof. Assume that $\Psi \Vert\!\models X$ and $\Phi \cup \{X\} \Vert\!\models Y$, but $\Psi \cup \Phi \Vert\!\not\models Y$. If follows that there exists an admissible partition $\mathsf{P} = \langle \mathsf{T}_{\mathsf{P}}, \mathsf{U}_{\mathsf{P}} \rangle$ such that $Z \cap \mathsf{T}_{\mathsf{P}} \neq \emptyset$ for each $Z \in \Psi \cup \Phi$, and $Y \cap \mathsf{T}_{\mathsf{P}} = \emptyset$. As $\Psi \Vert\!\models X$, we have $X \cap \mathsf{T}_{\mathsf{P}} \neq \emptyset$. But $\Phi \cup \{X\} \Vert\!\models Y$ and hence $Y \cap \mathsf{T}_{\mathsf{P}} \neq \emptyset$. A contradiction. □

As an immediate consequence of Proposition 6 we get:

Corollary 2. *If* $\Phi \Vert\!\models X$ *and* $\Phi \cup \{X\} \Vert\!\models Y$*, then* $\Phi \Vert\!\models Y$*.*

[3] A wff A is *valid* iff $A \in \mathsf{T}_{\mathsf{P}}$ for each admissible partition $\mathsf{P} = \langle \mathsf{T}_{\mathsf{P}}, \mathsf{U}_{\mathsf{P}} \rangle$.

3.3. Generalized entailment versus entailment and mc-entailment

Both entailment and mc-entailment are definable in terms of g-entailment. However, we need an auxiliary concept.

Definition 5. $\breve{X} =_{\text{df}} \{\{A\} : A \in X\}$.

$\breve{X}$ is thus the family of singleton sets based on the elements of X. The family $\breve{X}$ may be called the *disperse* of set X. Observe that $\breve{\emptyset} = \emptyset$.

The following holds:

Corollary 3. *For every admissible partition* $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$*:* $X \subset \mathsf{T_P}$ *iff for each* $Z \in \breve{X}$*:* $Z \subset \mathsf{T_P}$*.*

Proof. Just notice that $X \subset \mathsf{T_P}$ iff $B \in \mathsf{T_P}$, for any $B \in X$. □

By Corollary 3 we obtain that g-entailment and entailment are linked in the way described by:

Proposition 7. $X \models A$ *iff* $\breve{X} \;|\!|\!\models \{A\}$*.*

As for mc-entailment, again by Corollary 3, we have:

Proposition 8. $X \,|\!\models Y$ *iff* $\breve{X} \;|\!|\!\models Y$*.*

3.4. Generalized entailment and choices

Let us come back to Example 4 presented in Section 3.1. As we remarked, the following holds:

$$\{\{p \vee q \to s \vee r, p \vee q \to s \vee t\}, \{p, q\}\} \;|\!|\!\models \{s, r, t\}.$$

For brevity, we designate $\{p \vee q \to s \vee r, p \vee q \to s \vee t\}$ by X_1, $\{p, q\}$ by X_2, and $\{s, r, t\}$ by Y. Thus $\{X_1, X_2\} \;|\!|\!\models Y$. Now let us consider the following sets of wffs:

$$\begin{aligned} Z_1 &= \{p \vee q \to s \vee r, p\}, \\ Z_2 &= \{p \vee q \to s \vee r, q\}, \\ Z_3 &= \{p \vee q \to s \vee t, p\}, \\ Z_4 &= \{p \vee q \to s \vee t, q\}. \end{aligned}$$

Each Z_i, where $1 \leqslant i \leqslant 4$, is a set that contains exactly one representative of X_1 and exactly one representative of X_2. Observe that we have:

$$Z_i \,|\!\models Y$$

that is, each Z_i ($1 \leqslant i \leqslant 4$) mc-entails Y. In other words, any set which contains exactly one representative of X_1 and exactly one representative of X_2 mc-entails Y.

The above observation can be generalized and then turned into an equivalence, but some caution is needed. We have to express in exact terms the idea of a set which contains *exactly one representative* of each set belonging to a previously given family of sets. This can be done in many ways. In the next section we present a solution which, additionally, will be used in defining the second central concept of this paper, namely constructive generalized entailment.

3.4.1. $\chi^{\otimes}(\Phi)$-sets and $\chi(\Phi)$-sets

We introduce, first, the following technical concept[4] ($\times$ stands here for the sign of Cartesian product):

DEFINITION 6.
$$X^{\otimes} = \begin{cases} X \times \{X\} \text{ if } X \neq \emptyset, \\ \emptyset \text{ if } X = \emptyset. \end{cases}$$

Clearly, we have:

COROLLARY 4. *If $X \neq Z$, then $X^{\otimes} \cap Z^{\otimes} = \emptyset$.*

DEFINITION 7. $\Phi^{\otimes} =_{\mathrm{df}} \{X^{\otimes} : X \in \Phi\}$.

Obviously, if $\Phi = \emptyset$, then $\Phi^{\otimes} = \emptyset$. (To see this it suffices to observe that $\{X^{\otimes} : X \in \emptyset\} = \emptyset$.) The following holds:

COROLLARY 5. *If $\Phi^{\otimes} \neq \emptyset$ and $\emptyset \notin \Phi^{\otimes}$, then there exists a set γ such that γ comprises exactly one element $\langle A, X\rangle$ of each $X^{\otimes} \in \Phi^{\otimes}$.*

PROOF. By the Axiom of Choice (observe that Corollary 4 warrants that the elements of $\Phi^{\otimes}$ are pairwise disjoint). □

Our second technical concept is given by:

DEFINITION 8. γ is a $\chi^{\otimes}(\Phi)$-set iff

1. $\gamma \subset \bigcup \Phi^{\otimes}$ and
2. for each $X^{\otimes} \in \Phi^{\otimes}$ such that $X^{\otimes} \neq \emptyset$ there exists exactly one $\langle A, X\rangle \in X^{\otimes}$ such that $\langle A, X\rangle \in \gamma$.

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[4] I am indebted to Jerzy Pogonowski for his suggestion to use the concept for the purposes of this paper.

One can prove (the proof is given in Appendix):

Proposition 9. *For each family of sets* Φ *there exists a* $\chi^{\otimes}(\Phi)$*-set.*

Definition 9. Let γ be a $\chi^{\otimes}(\Phi)$-set.

$$\gamma^1 =_{\mathrm{df}} \{A : \langle A, X\rangle \in \gamma\}.$$

Now we are able to introduce:

Definition 10. Z is a $\chi(\Phi)$-set iff $Z = \gamma^1$ for some $\chi^{\otimes}(\Phi)$-set γ.

A $\chi(\Phi)$-set is a set comprising exactly one *representative* of each non-empty set belonging to Φ. One should not confuse the existence of exactly one representative of each set belonging to a family of sets with the existence of a system of distinct representatives of the family.[5] The representatives of distinct sets in a χ-set need not be distinct.

Example 6. Let $\Phi = \{X_1, X_2\}$, where $X_1 = \{p, q\}$ and $X_2 = \{p, r\}$. The following are $\chi^{\otimes}$-sets: $\{\langle p, X_1\rangle, \langle p, X_2\rangle\}$, $\{\langle p, X_1\rangle, \langle r, X_2\rangle\}$, $\{\langle q, X_1\rangle, \langle p, X_2\rangle\}$, $\{\langle q, X_1\rangle, \langle r, X_2\rangle\}$. Thus the family of $\chi(\Phi)$-sets comprises: $\{p\}$, $\{p, r\}$, $\{p, q\}$, $\{q, r\}$. As for the $\chi(\Phi)$-set $\{p\}$, p is the representative of X_1 and is the representative of X_2.

In the light of Proposition 9, the following holds:

Proposition 10. *For each family of sets* Φ *there exists at least one* $\chi(\Phi)$*-set.*

Let us also note (the proof is given in Appendix):

Proposition 11. *Let* $A \in X$ *for some* $X \in \Phi$*. There exists at least one* $\chi(\Phi)$*-set such that* A *belongs to this set.*

3.4.2. Generalized entailment and $\chi(\Phi)$-sets

A $\chi(\Phi)$-set can be intuitively understood as a "choice set": we choose from each set in Φ its representative. Thus quantifying over all $\chi(\Phi)$-sets amounts to quantifying over all possible choices of this kind. In this section we show that, for any family of non-empty sets of wffs, being g-entailed by the family amounts to being mc-entailed by each "choice set" associated with the family, that is, by any χ-set of the family.

Theorem 1. *Let* $\emptyset \notin \Phi$*. Then* $\Phi \mathrel{|\!|\!\models} Y$ *iff* $Z \mathrel{|\!\models} Y$ *for each* $\chi(\Phi)$*-set* Z*.*

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[5] As it is well-known, a system of distinct representatives — a transversal of a family of sets — does not always exist; cf. e.g. [23, Chapter 8].

Proof. ($\Rightarrow$) Assume that $\Phi \,||\!\models Y$. Suppose that that $Z \,||\!\not\models Y$ for some $\chi(\Phi)$-set Z. Thus there exists an admissible partition, P, such that $Z \subset \mathsf{T}_\mathsf{P}$ and $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$.

Assume that $Z = \emptyset$. Hence $\Phi = \emptyset$ or $\Phi = \{\emptyset\}$. However, by assumption $\emptyset \notin \Phi$. Thus $\Phi = \emptyset$. As $\Phi \,||\!\models Y$, by Proposition 4 we get that Y is a safeset. But $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$. A contradiction.

Now assume that $Z \neq \emptyset$. Since Z is a $\chi(\Phi)$-set, it contains elements of each set in Φ (recall that, by assumption, these sets are non-empty). Hence for each $X \in \Phi$ we have $X \cap \mathsf{T}_\mathsf{P} \neq \emptyset$. But $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$. It follows that $\Phi \,||\!\not\models Y$. A contradiction.

($\Leftarrow$) Assume that $\Phi \neq \emptyset$. Suppose that $\Phi \,||\!\not\models Y$. By assumption, $\emptyset \notin \Phi$. Thus for some admissible partition, P, we have $X \cap \mathsf{T}_\mathsf{P} \neq \emptyset$ for each $X \in \Phi$, and $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$. Recall that, again by assumption, Φ comprises non-empty sets. We assign to each set $X \in \Phi$ the corresponding set X^* by:

$$X^* = X \cap \mathsf{T}_\mathsf{P}$$

Let Φ^* be the family of all X^*-sets defined in the above manner. By Proposition 9, Φ^* has a $\chi^\otimes(\Phi^*)$-set, say, δ. Observe that $\delta \neq \emptyset$ and $\delta^1 \subset \mathsf{T}_\mathsf{P}$. We define a set γ by:

$$\gamma = \{\langle A, X\rangle \in \Phi^\otimes : \langle A, X^*\rangle \in \delta\}$$

It is clear that γ is a $\chi^\otimes(\Phi)$-set. Moreover, $\gamma^1 = \delta^1$. So there exists a $\chi(\Phi)$-set, namely γ^1, such that $\gamma^1 \subset \mathsf{T}_\mathsf{P}$. Hence $Z \,||\!\not\models Y$ for some $\chi(\Phi)$-set Z.

Finally, assume that $\Phi = \emptyset$. It follows that $\emptyset$ is the only $\chi(\Phi)$-set. Suppose that $\Phi \,||\!\not\models Y$. Thus there exists an admissible partition, P, such that $Y \cap \mathsf{T}_\mathsf{P} = \emptyset$. Therefore $\emptyset \,||\!\not\models Y$. □

Remark that the assumption '$\emptyset \notin \Phi$' is a necessary one. As we have shown (cf. Proposition 5), a family of sets that includes the empty set g-entails any set of wffs.

4. Constructive generalized entailment

Let us, again, come back to Example 4 presented in Section 3.1. Let $\Phi = \{X_1, X_2\}$, where $X_1 = \{p \vee q \to s \vee r, p \vee q \to s \vee t\}$ and $X_2 = \{p, q\}$. We have $\Phi \,||\!\models \{s, r, t\}$. The respective $\chi(\Phi)$-sets are: $\{p \vee q \to s \vee r, p\}$, $\{p \vee q \to s \vee r, q\}$, $\{p \vee q \to s \vee t, p\}$, $\{p \vee q \to s \vee t, q\}$. Each $\chi(\Phi)$-set mc-entails the set $\{s, r, t\}$. But observe that no $\chi(\Phi)$-set entails a single wff in $\{s, r, t\}$!

However, we also have:

$$\Phi \;|\!|\!\models \{s \vee r, s \vee t\}.$$

In this case g-entailment is *constructive*: for each $\chi(\Phi)$-set there exists a single wff in $\{s \vee r, s \vee t\}$ which is entailed by the $\chi(\Phi)$-set. Here is another example of this kind. Let $\Psi = \Phi \cup \{\neg s\}$. We have $\Psi \;|\!|\!\models \{r, t\}$. Each $\chi(\Psi)$-set results from a $\chi(\Phi)$-set by adding $\neg s$. It is easily visible that any $\chi(\Psi)$-set either entails r or entails t.

Constructive generalized entailment, or cg-entailment for short, is a relation between a family of sets of wffs and a set of wffs. We use $|\!\!\trianglerighteq$ as the symbol for cg-entailment.

Definition 11 (Cg-entailment). $\Phi \,|\!\!\trianglerighteq Y$ iff for each $\chi(\Phi)$-set X there exists $A \in Y$ such that $X \models A$.

Thus $\Phi \,|\!\!\trianglerighteq Y$ holds just in case each $\chi(\Phi)$-set entails some wff in Y. Remark that different $\chi(\Phi)$-sets may entail different elements of Y.

By Definition 1 we get:

Corollary 6. *$\Phi \,|\!\!\trianglerighteq Y$ iff for each $\chi(\Phi)$-set X there exists $A \in Y$ such that for any admissible partition $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ the following condition holds*:

- *if $X \subset \mathsf{T_P}$, then $A \in \mathsf{T_P}$.*

Clearly, we have:

Corollary 7. *If $\Phi \,|\!\!\trianglerighteq Y$, then $Y \neq \emptyset$.*

Observe that being a safeset is not sufficient for being cg-entailed by a family of wffs. For instance, $\{p, \neg p\}$ is a safeset (w.r.t. CPL), but $\{p, \neg p\}$ is not cg-entailed by the singleton family $\{\{p \vee \neg p\}\}$ (again, in CPL). The situation is different in the case of g-entailment (cf. Corollary 1).

4.1. Basic properties of constructive generalized entailment

Similarly as g-entailment, also cg-entailment behaves in a "consequence-like" manner.

Proposition 12. *$\Phi \,|\!\!\trianglerighteq Y$ for each $Y \in \Phi$ such that $Y \neq \emptyset$.*

PROOF. It suffices to observe that if $Y \in \Phi$ and $Y \neq \emptyset$, then each $\chi(\Phi)$-set contains an element of Y. □

PROPOSITION 13. *If* $\Phi \mathrel{|\!\!\gtrdot} Y$ *and* $\Phi \subset \Psi$, *then* $\Psi \mathrel{|\!\!\gtrdot} Y$.

PROOF. It suffices to observe that if $\Phi \subset \Psi$, then each $\chi(\Psi)$-set has a subset being a $\chi(\Phi)$-set. □

PROPOSITION 14. *If* $\Phi \mathrel{|\!\!\gtrdot} Y$ *and* $\Psi \mathrel{|\!\!\gtrdot} X$ *for each* $X \in \Phi$, *then* $\Psi \mathrel{|\!\!\gtrdot} Y$.

PROOF. Let W be an arbitrary but fixed $\chi(\Psi)$-set. By assumption, $\Psi \mathrel{|\!\!\gtrdot} X$ for each $X \in \Phi$. Thus for any $X \in \Phi$, the set $X_{(W)}$ defined by:

$$X_{(W)} = \{B \in X : W \models B\}$$

is non-empty. Let $\Phi_{(W)}$ be the family of all $X_{(W)}$-sets defined in the above manner. By Proposition 9, the family $\Phi_{(W)}$ has a $\chi^{\otimes}(\Phi_{(W)})$-set, say, μ. Thus μ^1 is a $\chi(\Phi_{(W)})$-set. Moreover, we have $W \models D$ for each $D \in \mu^1$.

We define:

$$\theta = \{\langle C, X\rangle \in \Phi^{\otimes} : \langle C, X_{(W)}\rangle \in \mu\}.$$

Clearly, θ is a $\chi^{\otimes}(\Phi)$-set and thus θ^1 is a $\chi(\Phi)$-set. Observe that $\theta^1 = \mu^1$. As $\Phi \mathrel{|\!\!\gtrdot} Y$, there exists $A \in Y$ such that $\theta^1 \models A$. But, since $\theta^1 = \mu^1$, we have $W \models D$ for each $D \in \theta^1$. Therefore $W \models A$. Hence $\Psi \mathrel{|\!\!\gtrdot} Y$. □

Note, however, that if $\emptyset \in \Phi$, then, by Corollary 7, it is not the case that $\Phi \mathrel{|\!\!\gtrdot} \emptyset$.

4.1.1. Constructive generalized entailment and cut

It should be noted that cg-entailment, similarly as g-entailment, has a feature analogous to cut.

PROPOSITION 15. *If* $\Psi \mathrel{|\!\!\gtrdot} X$ *and* $\Phi \cup \{X\} \mathrel{|\!\!\gtrdot} Y$, *then* $\Psi \cup \Phi \mathrel{|\!\!\gtrdot} Y$.

PROOF. Assume that $X \in \Phi$. Since $\Phi \cup \{X\} \mathrel{|\!\!\gtrdot} Y$ holds, we get $\Psi \cup \Phi \mathrel{|\!\!\gtrdot} Y$ by Proposition 13.

Assume that $X \notin \Phi$. Let Z be an arbitrary but fixed $\chi(\Psi \cup \Phi)$-set. Thus $Z = \gamma^1$ for some $\chi^{\otimes}(\Psi \cup \Phi)$-set γ. We define the following sets:

$$\gamma_\Psi = \{\langle C, W\rangle \in \gamma : W \in \Psi\}$$
$$\gamma_\Phi = \{\langle C, W\rangle \in \gamma : W \in \Phi\}$$

γ_Ψ is a $\chi^{\otimes}(\Psi)$-set, and γ_Φ is a $\chi^{\otimes}(\Phi)$-set. Thus $(\gamma_\Psi)^1$ is a $\chi(\Psi)$-set, and $(\gamma_\Phi)^1$ is a $\chi(\Phi)$-set.

Since, by assumption, $\Psi \mathrel{|\!\!\trianglerighteq} X$, there exists an element of X, say, A, such that $(\gamma_\Psi)^1 \models A$. Let us define:

$$\gamma^*_\Phi = \gamma_\Phi \cup \{\langle A, X\rangle\}$$

γ^*_Φ is a $\chi^\otimes(\Phi \cup \{X\})$-set (since $X \notin \Phi$) and thus $(\gamma^*_\Phi)^1$ is a $\chi(\Phi \cup \{X\})$-set. Clearly, $A \in (\gamma^*_\Phi)^1$. By assumption, $\Phi \cup \{X\} \mathrel{|\!\!\trianglerighteq} Y$, and hence $(\gamma^*_\Phi)^1 \models B$ for some $B \in Y$. On the other hand, we have:

$$(\gamma^*_\Phi)^1 = (\gamma_\Phi)^1 \cup \{A\}$$

and $(\gamma_\Psi)^1 \models A$. Hence $(\gamma_\Psi)^1 \cup (\gamma_\Phi)^1 \models B$. Since $((\gamma_\Psi)^1 \cup (\gamma_\Phi)^1) \subset Z$, it follows that $Z \models B$. Therefore $\Psi \cup \Phi \mathrel{|\!\!\trianglerighteq} Y$. □

As an immediate consequence of Proposition 15 we get:

Corollary 8. *$\Phi \mathrel{|\!\!\trianglerighteq} X$ and $\Phi \cup \{X\} \mathrel{|\!\!\trianglerighteq} Y$, then $\Phi \mathrel{|\!\!\trianglerighteq} Y$.*

4.2. Constructive generalized entailment versus entailment and generalized entailment

One can prove that entailment of a wff A from a set of wffs X amounts to cg-entailment of the singleton set $\{A\}$ from the disperse of X.

Proposition 16. *$X \models A$ iff $\breve{X} \mathrel{|\!\!\trianglerighteq} \{A\}$.*

Proof. By Corollary 3. Observe that X is the only $\chi(\breve{X})$-set. □

Let us now prove that cg-entailment is a special case of g-entailment.

Proposition 17. *If $\Phi \mathrel{|\!\!\trianglerighteq} Y$, then $\Phi \mathrel{|\!|\!\models} Y$.*

Proof. Assume that $\Phi \mathrel{|\!\!\trianglerighteq} Y$.

Let $\emptyset \notin \Phi$. Suppose that $\Phi \mathrel{|\!|\!\not\models} Y$. By Theorem 1, there exists a $\chi(\Phi)$-set, say, Z, such that $Z \mathrel{|\!\not\models} Y$. It follows that there is no $A \in Y$ such that $Z \models A$ and hence it is not the case that $\Phi \mathrel{|\!\!\trianglerighteq} Y$. A contradiction.

Let $\emptyset \in \Phi$. Thus, by Proposition 5, $\Phi \mathrel{|\!|\!\models} Y$. □

Note that the converse of Proposition 17 does not hold. The example presented at the beginning of Section 4 illustrates this. Here is another. We have:

$$\{\{p \vee q\}\} \mathrel{|\!|\!\models} \{p, q\},$$

but we do not have:

$$\{\{p \vee q\}\} \mathrel{|\!\!\trianglerighteq} \{p, q\}.$$

A more sophisticated counterexample is: we have $\{\emptyset\} \mathrel{||}\!\models \{p, \neg p\}$ (as $\{\emptyset\}$ g-entails every set), but we do not have $\{\emptyset\} \mathrel{|}\!\!> \{p, \neg p\}$ (since $\emptyset$ is the only $\chi(\{\emptyset\})$-set, and neither p nor $\neg p$ is entailed by $\emptyset$).

However, cg-entailment and g-entailment coincide on singleton sets provided that $\emptyset \notin \Phi$.

COROLLARY 9. *Let* $\emptyset \notin \Phi$. *Then* $\Phi \mathrel{||}\!\models \{A\}$ *iff* $\Phi \mathrel{|}\!\!> \{A\}$.

PROOF. By Theorem 1 and the fact that mc-entailment of $\{A\}$ and entailment of A coincide. □

5. Constructive generalized entailment and questions

Cg-entailment is a relation between a family of sets of wffs and a set of wffs. One of the possible ways of thinking about non-singleton sets of wffs in the context of cg-entailment it to regard them as *sets of direct answers to questions.*

5.1. Direct answers

In this section we follow an idea which is present in some, but not all logical theories of questions. The idea is: a question offers a set of alternatives and the alternatives are expressed by direct answers to the question. Direct answers, in turn, are these possible answers to a question that are "optimal" in the sense that they provide information of the required kind and, at the same time, provide neither more nor less information than it is requested.[6] Being true *is not* a prerequisite of being a direct answer.

Questions can be incorporated into formal languages in many ways (for overviews, see e.g. [11, 10, 32]). One option is to supplement the vocabulary of an already given formal language with some question/interrogative forming expressions. As an outcome one gets a formal language which involves (at least) two categories of formulas: declarative well-formed formulas (hereafter: d-wffs), which are supposed to represent declaratives, and erotetic formulas[7], which, in turn, are supposed to rep-

[6] As David Harrah (who introduced the concept) puts it, a direct answer: "*gives exactly what the question calls for.* (...) *The label 'direct'* (...) *connotes both logical sufficiency and immediacy*" ([11, p. 1]). According to Belnap, direct answers: "*are directly and precisely responsive to the question, giving neither more nor less information than what is called for*" ([1, p. 124]).

[7] After the Greek word 'erotema', which means 'question.'

resent questions. For transparency, in what follows we will call erotetic formulas simply questions. At the next step an assignment is made: once we have a question (of a formal language), we assign to it a unique set d-wffs (of the language). The d-wffs assigned to a question are called *direct answers* to the question. Details of the assignment are unimportant for the purposes of this paper. Depending on preferences, one can, for instance, follow Belnap's theory (cf. [2]), or Kubiński's theory (cf. [15]), or the semi-reductionistic approach assumed in Inferential Erotetic Logic (cf. e.g. [26, Chapter 3] or [30, Chapter 2]). What is important, however, is that direct answers are thought of as the "optimal" possible answers to questions considered.

Some specific assumptions. For simplicity, we assume that the empty set does not constitute the set of direct answers to any question. Moreover, we assume that each direct answer to a question is a d-wff, and that sets of direct answers to questions are not singletons (thus any set of direct answers contains at least two d-wffs).

As for semantics, we will be assuming that the semantic concepts introduced in Section 2 pertain to d-wffs.

Notation. The letter Q, with of without a subscript or a superscript, is a metalanguage variable for questions. We write $\mathbf{d}Q$ for the set of direct answers to question Q.

5.2. Constructive generalized entailment and interrogative entailment

Let us, first, introduce the following auxiliary notion.

Definition 12. Let Δ be a set of questions.

$$\mathbf{d}\Delta = \{Z : Z = \mathbf{d}Q^* \text{ for some } Q^* \in \Delta\}.$$

By $\mathbf{d}\Delta$ we thus mean the family of sets of direct answers to questions belonging to a set of questions Δ.

Consider the following condition:

$$\mathbf{d}\Delta \mathrel{|\!\!>} \mathbf{d}Q \tag{2}$$

The condition (2) is fulfilled just in case each "choice set" associated with the family $\mathbf{d}\Delta$ (more precisely, each $\chi(\mathbf{d}\Delta)$-set) entails some direct answer to question Q. There are many "choice sets" of this kind (as each question has at least two direct answers), and (2) quantifies over all such sets. Thus, generally speaking, the claim of condition is: whatever direct answers to questions of Δ you would get (but one to each of them), Q is answered, in a way depending on the answers just got. Hence (2) expresses, in the current conceptual setting, one of the intuitions connected with "interrogative entailment."[8]

Now consider the condition:

$$\mathbf{d}\Delta \cup \breve{X} \models\!\!\!\!> \mathbf{d}Q \tag{3}$$

Observe that any $\chi(\mathbf{d}\Delta \cup \breve{X})$-set contains all the d-wffs that belong to X. Moreover, it contains exactly one representative of every set of direct answers to questions that belong to Δ. Hence the intuitive content of the condition (3) is: whatever direct answers to questions of Δ you would get (but one to each of them), Q is answered, on the basis of X, yet in a way depending on the answers just got. Thus, (3) also expresses an intuition connected with "interrogative entailment", namely interrogative entailment based on a set of d-wffs.

Observe that (2) is a special case of (3), as $\breve{\emptyset} = \emptyset$. Interrogative entailment can thus be defined uniformly by:

Definition 13 (Interrogative entailment). Let Δ be a set of questions and X be a set of d-wffs.

$$\Delta, X \models Q \text{ iff } \mathbf{d}\Delta \cup \breve{X} \models\!\!\!\!> \mathbf{d}Q.$$

When $X = \emptyset$, we write $\Delta \models Q$ (recall that $\breve{\emptyset} = \emptyset$).

Finally, let us consider the conditions:

$$\mathbf{d}\Delta \models\!\!\!\!> \{A\}, \tag{4}$$

$$\mathbf{d}\Delta \cup \breve{X} \models\!\!\!\!> \{A\}, \tag{5}$$

$$\breve{X} \models\!\!\!\!> \mathbf{d}Q. \tag{6}$$

The intuitive content of condition (4) is: whatever direct answers to questions of Δ you would get, A is the case. Condition (5) expresses something like: whatever direct answers to questions of Δ you would get, A is the case assuming that X consists of truths. Condition (6) can be read: X resolves Q.

[8] Compare the following quotation from Groenendijk and Stokhof [10]: "*interrogatives $?\phi_1 \ldots ?\phi_n$ entail (...) interrogative $?\psi$ in a model $\boldsymbol{M}$ iff any proposition which completely answers all of the $?\phi_1 \ldots ?\phi_n$ in $\boldsymbol{M}$, also completely answers $?\psi$ in $\boldsymbol{M}$. Logical entailment amounts to entailment in all models.*" ([10, p.1090]).

5.3. Interrogative entailment and reducibility of questions to sets of questions

Interrogative entailment, explicated in terms of cg-entailment, is closely connected with the concepts of reducibility of questions to sets of questions introduced in Inferential Erotetic Logic. Definitions presented below are taken from [24], published in 1994.

Definition 14 (Reducibility). A question Q is reducible to a non-empty set of questions Δ iff

1. for each direct answer A to Q, for each question Q^* of Δ: A mc-entails the set of direct answers to Q^*,
2. each set made up of direct answers to the questions of Δ which contains exactly one direct answer to each question of Δ entails some direct answer to Q, and
3. no question in Δ has more direct answers than Q.

Sets made up of direct answers to all the questions of Δ which contain exactly one direct answer to each question of Δ can be identified with $\chi(\mathbf{d}\Delta)$-sets.[9] Given this, the content of clause (2) of the above definition can be expressed by:

$$\mathbf{d}\Delta \mid\!\!\gg \mathbf{d}Q$$

or equivalently by:

$$\Delta \models Q\,.$$

Thus reducibility of a question Q to a set of questions Δ yields interrogative entailment of the question Q from the set of questions Δ.

Reducibility based on a non-empty set of d-wffs is defined by:

Definition 15 (Generalized reducibility). A question Q is reducible to a non-empty set of questions Δ on the basis of a non-empty set of d-wffs X iff

1. for each direct answer A to Q and for each question Q^* of Δ, the set $X \cup \{A\}$ mc-entails the set of direct answers to Q^*, and
2. for each set Y made up of direct answers to the questions of Δ which contains exactly one direct answer to each question of Δ there is a

[9] More precisely, any (set made up of direct answers to the questions of Δ) which contains exactly one direct answer to each question of Δ stands in a 1-1 correspondence with some $\chi^{\otimes}(\mathbf{d}\Delta)$-set, and the family of $\chi^{\otimes}(\mathbf{d}\Delta)$-sets determines the family of $\chi(\mathbf{d}\Delta)$-sets.

direct answer B to Q such that B is entailed by the set $X \cup Y$, but B is not entailed by the set X alone, and
3. no question in Δ has more direct answers than Q.

For reasons analogous to these presented above we may conclude that reducibility of a question Q, on the basis of a non-empty set of d-wffs X, to a set of questions Δ yields interrogative entailment of Q from Δ enriched with X. Thus results concerning reducibility (generalized or not) shed some light on interrogative entailment understood in the sense of Definition 13, and provide information about cg-entailment in general.[10]

5.4. Constructive generalized entailment versus entailment in inquisitive semantics

In this section we point out some affinities between cg-entailment and the concept of entailment elaborated on within inquisitive semantics. Inquisitive semantics conceives entailment as preservation of *support*. Support, in turn, is a relation between an information state and a formula. At the propositional level (the only one which will interest us here) an information state is identified with a set of possible worlds. We will concentrate upon the most basic system of inquisitive semantics, labelled InqB, in its current version presented by Ivano Ciardelli in [7] and [8].[11]

The language of InqB is a propositional language over a countable set of propositional variables. The primitive logical constants are: $\bot$, $\wedge$, $\rightarrow$, and $\mathbin{\vee\!\!\!\vee}$; the latter is called *inquisitive disjunction*. The set of wffs is defined in the standard manner. A wff of the form $A \mathbin{\vee\!\!\!\vee} B$ can be intuitively thought of as a question *whether* A *or* B. Disjunction, $\vee$, and negation, $\neg$, are defined by:

$$\neg A =_{\mathrm{df}} A \rightarrow \bot$$
$$A \vee B =_{\mathrm{df}} \neg(\neg A \wedge \neg B)$$

A wff is *classical* if it does not involve an occurrence of $\mathbin{\vee\!\!\!\vee}$.

[10] These results are presented in [24, 13, 14, 29].

[11] This system differs somehow, syntactically and semantically, from the system presented previously under the same label (cf. e.g. [6]). The language of the “old” version of propositional InqB contained only inquisitive disjunction, while in the “new” version classical disjunction occurs as well (as a defined connective). Moreover, these versions differ in conceptualizing questions. The basics of the semantics of the “new” version are much in line proposed in [31].

A *model* is a pair $\langle \mathcal{W}, V\rangle$, where $\mathcal{W}$ is a non-empty set of possible worlds and V is a valuation function that assigns to a wff and a world in $\mathcal{W}$ either truth, $\mathbf{1}$, or falsity, $\mathbf{0}$. A *state* of a model $\mathcal{M} = \langle \mathcal{W}, V\rangle$ is a subset of $\mathcal{W}$; states of a model $\mathcal{M}$ will be called below $\mathcal{M}$-*states*. Let σ be a $\mathcal{M}$-state. The relation of *support*, $\succ$, is defined by (we use $\mathtt{p}$ as a metalanguage variable for propositional variables):

Definition 16 (Support). 1. $\sigma \succ \mathtt{p}$ iff $V(\mathtt{p}, w) = \mathbf{1}$ for all $w \in \sigma$,
2. $\sigma \succ \bot$ iff $\sigma = \emptyset$,
3. $\sigma \succ A \wedge B$ iff $\sigma \succ A$ and $\sigma \succ B$,
4. $\sigma \succ A \rightarrow B$ iff for all $\tau \subset \sigma$: if $\tau \succ A$, then $\tau \succ B$,
5. $\sigma \succ A \mathbin{\vee\!\!\!\vee} B$ iff $\sigma \succ A$ or $\sigma \succ B$.

Truth of a wff in a world w of a model is defined as support by the singleton set $\{w\}$.

As for classical wffs, support by a $\mathcal{M}$-state σ and truth in each world of the state σ coincide. However, this is not so for wffs which contain inquisitive disjunction. For example, let $\mathcal{M} = \langle \mathcal{W}, V\rangle$ be a model such that for some $w_1, w_2 \in \mathcal{W}$ we have: $V(p, w_1) = \mathbf{1}$, $V(q, w_1) = \mathbf{0}$, $V(p, w_2) = \mathbf{0}$, and $V(q, w_2) = \mathbf{1}$. Clearly, $p \mathbin{\vee\!\!\!\vee} q$ is true in each world of the $\mathcal{M}$-state $\{w_1, w_2\}$, but neither p nor q is supported by $\{w_1, w_2\}$, and thus $\{w_1, w_2\}$ does not support $p \mathbin{\vee\!\!\!\vee} q$.

InqB distinguishes between statements and questions at the semantic level. As for the current version of InqB, a wff A is a *statement* just in case it is truth-conditional, that is, for any model $\mathcal{M}$ and any $\mathcal{M}$-state σ, A is supported by σ iff A is true in each world of σ. Classical wffs are all statements. *Questions* are those wffs which are not truth-conditional. Let us stress that the syntax of InqB allows for conjoining questions and statements by means of connectives.

Entailment in InqB, $\models_{\mathsf{InqB}}$, is defined by:

Definition 17. $X \models_{\mathsf{InqB}} A$ iff for each model $\mathcal{M}$ and each $\mathcal{M}$-state σ: if $\sigma \succ C$ for all $C \in X$, then $\sigma \succ A$.

Note that X can involve questions, and A can be a question. As for statements, $\models_{\mathsf{InqB}}$ boils down to CPL-entailment.

Proposition 18. *Let X be a set of statements, and A be a statement. Then $X \models_{\mathsf{InqB}} A$ iff $X \models_{\mathsf{CPL}} A$.*

InqB effectively assigns to each wff a non-empty set of classical/truth-conditional wffs called its *resolutions*; the details of the assignment are unimportant for our purposes (cf. [7, p. 56]). Generally speaking, the resolution of a truth-conditional wff is a singleton set which contains the wff, while resolutions of questions have at least two elements. What is important, however, is the following fact (we write $\mathcal{R}(A)$ for the set of resolutions of A):

PROPOSITION 19 (Ciardelli [7]). *For any wff A, any model $\mathcal{M}$ and any $\mathcal{M}$-state σ: $\sigma \succ A$ iff $\sigma \succ C$ for some $C \in \mathcal{R}(A)$.*

A *resolution function* for a set of wffs X is a map f from X to the set of classical wffs such that for each $A \in X$ we have $f(A) \in \mathcal{R}(A)$. A set of wffs Y is a resolution of a set of wffs X iff Y is an image of X under some resolution function. We put:

$$\mathcal{R}(X) = \{f[X] : f \text{ is a resolution function for } X\}.$$

The following holds:

THEOREM 2 (Ciardelli [7]). *For any set of wffs X and any wff A: $X \models_{\mathsf{InqB}} A$ iff for every $Y \in \mathcal{R}(X)$ there is some $C \in \mathcal{R}(A)$ such that $Y \models_{\mathsf{InqB}} C$.*

As resolutions are classical wffs and any classical wff is a statement, Theorem 2 and Proposition 18 yield:

COROLLARY 10. *For any set of wffs X and any wff A: $X \models_{\mathsf{InqB}} A$ iff for every $Y \in \mathcal{R}(X)$ there is some $C \in \mathcal{R}(A)$ such that $Y \models_{\mathsf{CPL}} C$.*

One can prove that inquisitive entailment in InqB and cg-entailment based on CPL are closely connected. In order to show this let us first introduce the following notion:

$$\Phi_X^{\mathcal{R}} = \{Z : Z = \mathcal{R}(A) \text{ for some } A \in X\}.$$

$\Phi_X^{\mathcal{R}}$ is thus the family of resolution sets of a set of wffs X.

We need:

LEMMA 1. *$Y \in \mathcal{R}(X)$ iff Y is a $\chi(\Phi_X^{\mathcal{R}})$-set.*

PROOF. ($\Rightarrow$) Suppose that $Y \in \mathcal{R}(X)$. Then for some resolution function f for X we have $Y = f[X]$. Let us consider the following set:

$$\gamma = \{\langle \alpha, \beta \rangle : \alpha = f(A) \text{ and } \beta = \mathcal{R}(A), \text{ for some } A \in X\}$$

It is easily seen that γ is a $\chi^{\otimes}(\Phi_X^{\mathcal{R}})$-set. Thus γ^1 is a $\chi(\Phi_X^{\mathcal{R}})$-set and, obviously, $\gamma^1 = Y$.

($\Leftarrow$) Assume that Y is a $\chi(\Phi_X^{\mathcal{R}})$-set. Thus $Y = \gamma^1$ for some $\chi^{\otimes}(\Phi_X^{\mathcal{R}})$-set γ. Let us define the following function f^* from γ to the set of classical wffs:

$$f^*(\langle C, \mathcal{R}(A)\rangle) = C, \text{ where } \langle C, \mathcal{R}(A)\rangle \in \gamma.$$

Next, we define the following function f with the domain X:

$$f(A) = f^*(\langle C, \mathcal{R}(A)\rangle).$$

Clearly, f is a resolution function for X and Y is the image of X under f. □

Thus, by Corollary 10 and Lemma 1, the following is true:

Theorem 3. *$X \models_{\mathsf{InqB}} A$ iff $\Phi_X^{\mathcal{R}} \mathrel{|\!\!\gtrdot}_{\mathsf{CPL}} \mathcal{R}(A)$.*

Therefore a wff A (a statement or a question) is entailed in InqB by a set of wffs X (which may contain statements and/or questions) just in case the family of resolution sets of wffs in X cg-entails in CPL the resolution set of A.

Remark. The connection between inquisitive entailment and cg-entailment established in Theorem 3 rests heavily on a peculiar property of InqB, namely the existence of resolution sets for wffs. In general, resolution sets are supposed to be linked with wffs and support in the manner analogous to that characterized by Proposition 19. However, there exists inquisitive logics which lack resolution sets for wffs (for details, see [7]). Thus one cannot say that inquisitive entailment is always reducible to cg-entailment.

5.5. Constructive generalized entailment and e-scenarios

How can one establish which sets of wffs are cg-entailed by which families of sets of wffs? As for the classical propositional case, a solution is suggested by the content of Theorem 3: one can make use of proof systems for InqB, which enable either Hilbert-style proofs (cf. [5]) or proofs in the natural deduction format (cf. [7, 8]). The aim of this section is to show that a certain formal tool developed within Inferential Erotetic Logic, namely erotetic search scenarios (e-scenarios for short), can also be helpful. Although the concept of e-scenario was introduced for different purposes, e-scenarios carry, as we will show, information about concrete cases of cg-entailment.

5.6. What are e-scenarios?

For space reasons, we will not explain here intuitions which lie behind the concept of e-scenario. An interested reader is advised to consult, e.g., [27], [28], or [30, Part III]. We will provide here only the relevant definitions.

The general setting needed for defining e-scenarios is that described in Subsection 5.1 above. Thus we operate with a formal language which has (at least) two categories of wffs: declarative well-formed formulas (d-wffs) and questions. We assume some assignment of direct answers to questions. The "declarative part" of the language is supposed to be supplemented with a semantics rich enough to define the concepts of entailment, $\models$, and mc-entailment, $\parallel\!\!\models$, for d-wffs. As for the basic inferential relation for questions, we use the concept of erotetic implication, being one of the fundamental concepts of Inferential Erotetic Logic. Again, we provide here only the definition; for intuitions, see e.g. [25], or [30, chapters 5 and 7].

DEFINITION 18 (Erotetic implication). A question Q implies a question Q_1 on the basis of a set of d-wffs X (in symbols: $\mathbf{Im}(Q, X, Q_1)$) iff

1. for each $A \in \mathbf{d}Q : X \cup \{A\} \parallel\!\!\models \mathbf{d}Q_1$ and
2. for each $B \in \mathbf{d}Q_1$ there exists a non-empty proper subset Y of $\mathbf{d}Q$ such that $X \cup \{B\} \parallel\!\!\models Y$.

E-scenarios can be defined either as families of interconnected e-derivations or as labelled trees. In this paper we choose the first option. The definitions presented below come from [27], published in 2003.

DEFINITION 19 (E-derivation). A finite sequence $\mathbf{s} = \mathtt{s}_1, \ldots, \mathtt{s}_n$ of wffs is an erotetic derivation (e-derivation for short) of a direct answer A to question Q on the basis of a set of d-wffs X iff $\mathtt{s}_1 = Q$, $\mathtt{s}_n = A$, and the following conditions hold:

1. for each question $\mathtt{s}_k$ of $\mathbf{s}$ such that $k > 1$:
   (a) $\mathbf{d}\mathtt{s}_k \neq \mathbf{d}Q$,
   (b) $\mathtt{s}_k$ is erotetically implied by some question $\mathtt{s}_j$ which precedes $\mathtt{s}_k$ in $\mathbf{s}$ on the basis of the empty set, or on the basis of a non-empty set of d-wffs such that each element of this set precedes $\mathtt{s}_k$ in $\mathbf{s}$, and
   (c) $\mathtt{s}_{k+1}$ is either a direct answer to $\mathtt{s}_k$ or a question;
2. for each d-wff $\mathtt{s}_i$ of $\mathbf{s}$:

(a) $\mathbf{s}_i \in X$, or
(b) $\mathbf{s}_i$ is a direct answer to $\mathbf{s}_{i-1}$, where $\mathbf{s}_{i-1} \neq Q$, or
(c) $\mathbf{s}_i$ is entailed by some non-empty set of d-wffs such that each element of this set precedes $\mathbf{s}_i$ in $\mathbf{s}$;

Note that by "precedes" we do not mean "immediately precedes."

If $\mathbf{s} = \mathbf{s}_1, \ldots, \mathbf{s}_n$ is an e-derivation of a direct answer to question Q on the basis of X, each question of $\mathbf{s}$ different from Q is called an *auxiliary question* of $\mathbf{s}$.

DEFINITION 20 (Query of an e-derivation). A term $\mathbf{s}_k$ (where $1 < k < n$) of an e-derivation $\mathbf{s} = \mathbf{s}_1, \ldots, \mathbf{s}_n$ is a query of $\mathbf{s}$ if $\mathbf{s}_k$ is a question and $\mathbf{s}_{k+1}$ is a direct answer to $\mathbf{s}_k$.

Queries are thus defined syntactically. Note that an e-derivation can involve auxiliary questions that are not queries. Generally speaking, auxiliary questions that are not queries facilitate erotetic implication of queries. The immediate successors of queries are direct answers to them.

An e-scenario is a set of interconnected e-derivations.

DEFINITION 21 (E-scenario). A finite family Σ of sequences of wffs is an erotetic search scenario (e-scenario for short) for a question Q relative to a set of d-wffs X iff each element of Σ is an e-derivation of a direct answer to Q on the basis of X and the following conditions hold:

1. $\mathsf{d}Q \cap X = \emptyset$;
2. Σ contains at least two elements;
3. for each element $\mathbf{s} = \mathbf{s}_1, \ldots, \mathbf{s}_n$ of Σ, for each index k, where $1 \leqslant k < n$:
   (a) if $\mathbf{s}_k$ is a question and $\mathbf{s}_{k+1}$ is a direct answer to $\mathbf{s}_k$, then for each direct answer B to $\mathbf{s}_k$: the family Σ contains a certain e-derivation $\mathbf{s}^* = \mathbf{s}^*_1, \mathbf{s}^*_2, \ldots, \mathbf{s}^*_m$ such that $\mathbf{s}_j = \mathbf{s}^*_j$ for $j = 1, \ldots, k$, and $\mathbf{s}^*_{k+1} = B$;
   (b) if $\mathbf{s}_k$ is a d-wff, or $\mathbf{s}_k$ is a question and $\mathbf{s}_{k+1}$ is not a direct answer to $\mathbf{s}_k$, then for each e-derivation $\mathbf{s}^* = \mathbf{s}^*_1, \mathbf{s}^*_2, \ldots, \mathbf{s}^*_m$ in Σ such that $\mathbf{s}_j = \mathbf{s}^*_j$ for $j = 1, \ldots, k$ we have $\mathbf{s}_{k+1} = \mathbf{s}^*_{k+1}$.

The e-derivations which are elements of an e-scenario Σ for Q relative to X will be called *paths* of Σ, the question Q will be called the *principal question* of Σ, and any other question of Σ is called an *auxiliary question* of the e-scenario.

Clause (3a) expresses the idea of *fairness with respect to queries*: if A is a direct answer to a query that immediately succeeds the query on a path $\mathbf{s}$ of an e-scenario Σ, then for *any* direct answer B to the query that is different from A there exists a path $\mathbf{s}^*$ of Σ which is identical with $\mathbf{s}$ to the level of the query, and then has B as the immediate successor of the query. Thus, roughly, for any path and any query on that path there exists a cluster of related paths which share the query and its predecessors, but diverge with respect to the direct answers to the query. Thus each direct answer to a query contributes to some path and thus to a derivation of an answer to the principal question: there are no "dead ends."

Clause (3b), in turn, expresses the idea of *regularity*: if $\mathtt{s}_k$ $(k < n)$ is a d-wff of a path $\mathbf{s} = \mathtt{s}_1, \ldots, \mathtt{s}_n$, or $\mathtt{s}_k$ is a question of $\mathbf{s}$ that is not a query, then each path which is identical with $\mathbf{s}$ to the level of $\mathtt{s}_k$ has the wff $\mathtt{s}_{k+1}$ as the $k+1$st term. In other words, d-wffs as well as questions that are not queries are "used" within a cluster of related paths in a uniform manner. Hence only queries are "branching points" of e-scenarios.

DEFINITION 22 (Query of e-scenario). A query of an e-scenario is a query of a path of the e-scenario.

In diagrams of e-scenarios queries are displayed as branching nodes.

Figures 1–3 present examples of e-scenarios. As for examples presented in figures 1 and 2, we use CPL as the underlying logic of d-wffs, while the third example is based on FOL. We write $?\{A_1, \ldots, A_n\}$ for a question whose set of direct answers is $\{A_1, \ldots, A_n\}$. For conciseness, we abbreviate $?\{A, \neg A\}$ as $?A$. A question whose set of direct answers is $\{A \wedge B, A \wedge \neg B, \neg A \wedge B, \neg A \wedge \neg B\}$ is concisely written as $? \pm |A, B|$. The letters p, q, r, s, t, u are propositional variables, the letters $\mathtt{P}, \mathtt{S}, \mathtt{T}, \mathtt{U}$ are one-place predicates, and $\mathtt{a}$ stands for individual constant. Recall that e-scenarios can involve auxiliary questions that are not queries.[12]

One can prove:

COROLLARY 11. 1. *Each query of an e-scenario is a question with a finite number of direct answers.*

2. *Each path of an e-scenario involves at least one query.*

[12] Auxiliary questions that are not queries play no pragmatic role, but are necessary to facilitate erotetic implication of queries. For example, in building the e-scenario displayed in Figure 3 we make use of the following facts concerning erotetic implication based on FOL: $\mathbf{Im}(?\mathtt{Pa}, \forall x(\mathtt{P}x \leftrightarrow \mathtt{S}x \vee (\mathtt{T}x \wedge \mathtt{U}x)), ? \pm |\mathtt{Sa}, \mathtt{Ta} \wedge \mathtt{Ua}|)$, $\mathbf{Im}(? \pm |\mathtt{Sa}, \mathtt{Ta} \wedge \mathtt{Ua}|, ?\mathtt{Sa})$, $\mathbf{Im}(? \pm |\mathtt{Sa}, \mathtt{Ta} \wedge \mathtt{Ua}|, ?(\mathtt{Ta} \wedge \mathtt{Ua}))$, $\mathbf{Im}(?(\mathtt{Ta} \wedge \mathtt{Ua}), ?\{\mathtt{Ta} \wedge \mathtt{Ua}, \mathtt{Ta} \wedge \neg\mathtt{Ua}, \neg\mathtt{Ta}\})$, $\mathbf{Im}(?\{\mathtt{Ta} \wedge \mathtt{Ua}, \mathtt{Ta} \wedge \neg\mathtt{Ua}, \neg\mathtt{Ta}\}, ?\mathtt{Ta})$, $\mathbf{Im}(?(\mathtt{Ta} \wedge \mathtt{Ua}), \mathtt{Ta}, ?\mathtt{Ua})$.

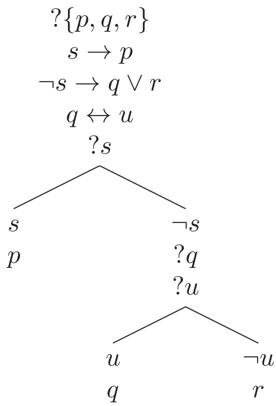

Figure 1.

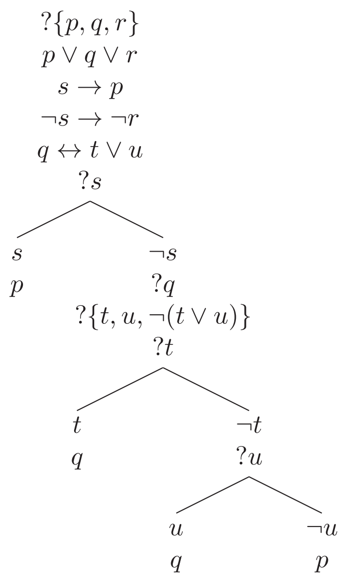

Figure 2.

Every e-scenario has the *first query*, shared by all the paths of the scenario. More precisely, the following holds:

Corollary 12. *Let Σ be an e-scenario for Q relative to X. There exist an index $k > 1$ and a question Q^* such that:*

1. *Q^* is the k-th term of every path of Σ,*

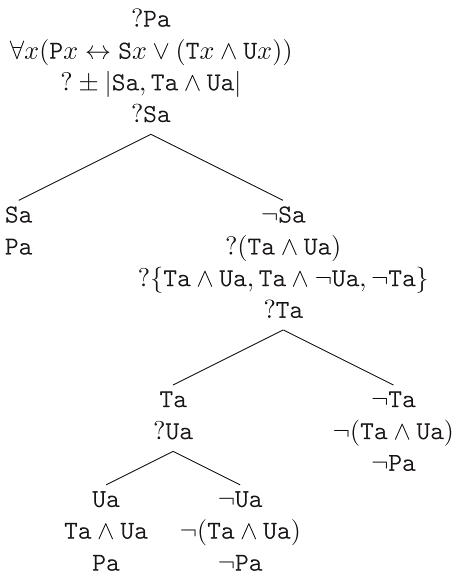

Figure 3.

2. *the k-th term of a path of Σ is a query of the path and hence of Σ,*
3. *k is the least index of a query of Σ.*

For proofs, see [30, Chapter 9].

5.7. Constructive generalized entailment in e-scenarios

E-scenarios are linked with cg-entailment in the way described by the following:

THEOREM 4. *Let Σ be an e-scenario for a question Q relative to a set of d-wffs X, and let Δ be the set of all queries of Σ. Then $\mathbf{d}\Delta \cup \check{X} \mathrel{|\!\!\triangleright} \mathbf{d}Q$.*

PROOF. It suffices to show that for each $\chi(\mathbf{d}\Delta)$-set Z there exists a direct answer A to Q such that $Z \cup X \models A$.[13] Let Z be an arbitrary but fixed $\chi(\mathbf{d}\Delta)$-set. Since the number of queries of an e-scenario is always

[13] Observe that a $\chi(\mathbf{d}\Delta \cup \check{X})$-set can be displayed as:

$$Z \cup X,$$

where Z is a $\chi(\mathbf{d}\Delta)$-set. More precisely, the following holds:

$$\{Y : Y \text{ is a } \chi(\mathbf{d}\Delta \cup \check{X})\text{-set}\} = \{Z \cup X : Z \text{ is a } \chi(\mathbf{d}\Delta)\text{-set}\}.$$

finite (as an e-scenario is a finite set of finite sequences of wffs), Z is a finite set. At the same time, due to Corollary 11, Z is non-empty.

By Corollary 11, Σ has the "first query". It follows that all the e-derivations/sequences in Σ share an initial sequent which can be displayed as ($'$ stands here for the concatenation-sign):

$$Q\,'\zeta_1\,'Q_1\,,$$

where ζ_1 is a sequence of d-wffs and/or auxiliary questions that are not queries of Σ (in some cases this sequence can be empty), and Q_1 is the "first" query of Σ. Direct answers to Q_1 are immediate successors of Q_1 at all the paths of Σ, and each direct answer to Q_1 performs this role. So some element of Z, say, B_1, is an immediate successor of Q_1 at some path(s) of Σ.

We act as follows.

First, we remove from Σ all the sequences/e-derivations which do not have B_1 as the immediate successor of Q_1. Let us designate the result of this operation by Σ_1.

Σ_1 is either a singleton set or has at least two elements (but a finite number of them, as e-scenarios are finite sets of e-derivations).

It can be easily shown that the following is true:

(†) for each path $\mathbf{s}$ of Σ we have:

$$X \cup \mathcal{A}(\mathbf{s}) \models D^{\mathbf{s}},$$

where $\mathcal{A}(\mathbf{s})$ comprises the terms of $\mathbf{s}$ which are immediate successors of queries of $\mathbf{s}$, that is, the direct answers to the queries which occur at the path, and $D^{\mathbf{s}}$ is the direct answer to Q which is the last term of $\mathbf{s}$.

Assume that Σ_1 is a singleton set, and that $\mathbf{s}^{(1)} \in \Sigma_1$. It follows that no other element of Δ (besides Q_1) occurs at $\mathbf{s}^{(1)}$, that is, Q_1 is the only query of $\mathbf{s}_1$. Thus $\mathcal{A}(\mathbf{s}^{(1)}) = \{B_1\}$. By (†) we get $X \cup \{B_1\} \models D^{\mathbf{s}^{(1)}}$, and hence $X \cup Z \models D^{\mathbf{s}^{(1)}}$ as required.

Now assume that Σ_1 is not a singleton set. In this case there exists $Q_2 \in \Delta$ such that all the sequences in Σ_1 share an initial path:

$$Q\,'\zeta_1\,'Q_1\,'B_1\,'\zeta_2\,'Q_2$$

Let $B_2 \in Z$, i.e., let B_2 be the direct answer to Q_2 that belongs to Z. We remove from Σ_1 all the paths/e-derivations that do not have B_2 as the immediate successor of Q_2. We arrive at a set of paths/e-derivations, Σ_2, which is either a singleton set or not.

If Σ_2 is a singleton set and $\mathbf{s}^{(2)} \in \Sigma_2$, we have $\mathcal{A}(\mathbf{s}^{(2)}) = \{B_1, B_2\}$. In this case we reason analogously as above, but with respect to $\mathbf{s}^{(2)}$.

If Σ_2 is not a singleton set, then, again, there exists $Q_3 \in \Delta$ such that all the sequences in Γ_2 share an initial sequent:

$$Q\,'\zeta_1\,'Q_1\,'B_1\,'\zeta_2\,'Q_2\,'B_2\,'\zeta_3\,'Q_3\,.$$

We take the direct answer to Q_3 that occurs in Z and act analogously as above with respect to the answer taken.

As each path of Σ is a finite sequence and thus involves only a finite number of occurrences of queries, it is clear that by acting in the above manner we will arrive, in a finite number of steps, at a singleton set of paths, say, $\{\mathbf{s}^{(*)}\}$. Clearly, $\mathcal{A}(\mathbf{s}^{(*)}) \subset Z$. Since, by (†), $X \cup \mathcal{A}(\mathbf{s}^{(*)}) \models D^{\mathbf{s}^{(*)}}$, it follows that $X \cup Z \models D^{\mathbf{s}^{(*)}}$. □

As an immediate consequence of Theorem 4 and Definition 13 we get:

Corollary 13. *Let Σ be an e-scenario for a question Q relative to a set of d-wffs X, and let Δ be the set of all queries of Σ. Then $\Delta, X \models Q$.*

In other words, the set of all queries of an e-scenario for a question Q relative to a set of d-wffs X interrogatively entails, on the basis of X, the question Q.

5.7.1. Examples and comments

Since Theorem 4 holds, the e-scenario displayed in Figure 3 shows that the following family of sets of wffs:

$$\{\{\forall x(\mathtt{P}x \leftrightarrow \mathtt{S}x \vee (\mathtt{T}x \wedge \mathtt{U}x))\}, \{\mathtt{Sa}, \neg\mathtt{Sa}\}, \{\mathtt{Ta}, \neg\mathtt{Ta}\}, \{\mathtt{Ua}, \neg\mathtt{Ua}\}\}$$

(for brevity, let us designate it by Ψ) cg-entails the set of wffs:

$$\{\mathtt{Pa}, \neg\mathtt{Pa}\} \tag{7}$$

This is the *global* information we get thanks to the existence of an appropriate e-scenario. It is non-trivial. Observe that (7) is a safeset. Recall that it is not the case that any safeset is cg-entailed by any family of sets of wffs. But the family Ψ cg-entails the safeset (7), which means that each "choice set" of Ψ (more precisely, each $\chi(\Psi)$-set) entails some wff in (7). However, the e-scenario presented in Figure 3 also provides us *more detailed* information, namely:

1. $\mathtt{Pa}$ is entailed by: (a) any $\chi(\Psi)$-set that contains $\mathtt{Sa}$, or (b) the $\chi(\Psi)$-set $\{\neg\mathtt{Sa}, \mathtt{Ta}, \mathtt{Ua}\}$;

– *Ask* ?$\mathtt{Sa}$ *first.*
– *If you receive the answer* $\mathtt{Sa}$*, the solution is* $\mathtt{Pa}$*. No further question is needed.*
– *If you receive the answer* $\neg\mathtt{Sa}$*, ask* ?$\mathtt{Ta}$*.*
– *If you receive the answer* $\neg\mathtt{Ta}$*, the solution is* $\neg\mathtt{Pa}$*. No further question is needed.*
– *If you receive the answer* $\mathtt{Ta}$*, ask* ?$\mathtt{Ua}$*.*
– *If you receive the answer* $\mathtt{Ua}$*, the solution is* $\mathtt{Pa}$*.*
– *If you receive the answer* $\neg\mathtt{Ua}$*, the solution is* $\neg\mathtt{Pa}$*.*

Figure 4. Knowledge-seeking procedure

2. $\neg\mathtt{Pa}$ is entailed by: (a) any $\chi(\Psi)$-set that contains $\neg\mathtt{Sa}$ and $\neg\mathtt{Ta}$, or (b) the $\chi(\Psi)$-set $\{\neg\mathtt{Sa}, \mathtt{Ta}, \neg\mathtt{Ua}\}$.

The other side of the same coin is: the e-scenario presented in Figure 3 witnesses the following case of interrogative entailment:

$$\{?\mathtt{Sa}, ?\mathtt{Ta}, ?\mathtt{Ua}\}, \{\forall x(\mathtt{P}x \leftrightarrow \mathtt{S}x \vee (\mathtt{T}x \wedge \mathtt{U}x))\} \models ?\mathtt{Pa}.$$

The *global* information is: whatever (direct) answers to questions: ?$\mathtt{Sa}$, ?$\mathtt{Ta}$, ?$\mathtt{Ua}$ you would get, question ?$\mathtt{Pa}$ is resolved, in a way dependant on the answers just got. But the e-scenario provides more information: it shows that in some cases(s) you do not need to ask all the questions from {?$\mathtt{Sa}$, ?$\mathtt{Ta}$, ?$\mathtt{Ua}$} in order to resolve the entailed question ?$\mathtt{Pa}$, but only some of them. In particular, the e-scenario displayed in Figure 3 justifies the *knowledge-seeking procedure* from Figure 4.

Now let us take a look at the e-scenario displayed in Figure 1. For brevity, we designate the set of wffs $\{s \to p, \neg s \to q \vee r, q \leftrightarrow u\}$ by Y. The e-scenario witnesses the following cases of cg-entailment and interrogative entailment:

$$\breve{Y} \cup \{\{s, \neg s\}, \{u, \neg u\}\} \mathrel{|\!\!\gtrdot} \{p, q, r\},$$
$$\{?s, ?u\}, Y \models ?\{p, q, r\}.$$

Observe that $\{p, q, r\}$ is not a safeset. The additional information carried by the e-scenario is, roughly, this. Any “choice set“ that contains s entails p and thus there is no need for asking $?u$ when s has been received. If, however, $\neg s$ is the case, then q holds provided that u holds; otherwise r holds.

Finally, the e-scenario presented in Figure 2 justifies the following claims (we designate the set $\{p \vee q \vee r, s \rightarrow p, \neg s \rightarrow \neg r, q \leftrightarrow t \vee u\}$ by Z);

$$\breve{Z} \cup \{\{s, \neg s\}, \{t, \neg t\}, \{u, \neg u\}\} \mathrel{|\!\!\!>} \{p, q, r\},$$
$$\{?s, ?t, ?u\}, Z \models ?\{p, q, r\}.$$

The due comments are similar as in the previous cases, but with one exception. Observe that there is no path of the e-scenario that leads to the answer r. On the other hand, the answers p and q can be reached in multiple ways.

As we have seen, an e-scenario carries global information concerning cg-entailment and interrogative entailment, but one can also extract from it some more detailed information useful for problem-solving. The latter effect is hardly surprising. Problem-solving was the initial, intended area of applicability of e-scenarios, and they seem quite useful in this field (cf. e.g. [30, Chapter 13] and [12]). However, it also occurred that e-scenarios are a powerful tool for modelling some aspects of dialogues (cf. e.g. [17]), including man-machine dialogue interactions (cf. [16, 18]). E-scenarios are linked with proof theory as well (see [22, 28]). Recently a dedicated software which enables, inter alia, automatic generation of e-scenarios out of a predefined set of e-scenarios, has been designed (cf. [4]). This, in view of Theorem 4, brings a computational perspective into research on cg-entailment and/or interrogative entailment.

Appendix

We need:

Corollary 14. *If $\Phi = \emptyset$, then $\emptyset$ is the only $\chi^{\otimes}(\Phi)$-set.*

Proof. If $\Phi = \emptyset$, then $\Phi^{\otimes} = \emptyset$. Hence clause (2) of Definition 8 is fulfilled by $\emptyset$ (since there is no $X^{\otimes} \in \Phi^{\otimes}$ such that $X^{\otimes} \neq \emptyset$). Clearly, $\bigcup \emptyset = \emptyset$, and $\emptyset$ is the only subset of $\bigcup \Phi^{\otimes}$. □

Corollary 15. *If $\Phi = \{\emptyset\}$, then $\emptyset$ is the only $\chi^{\otimes}(\Phi)$-set.*

Proof. $\emptyset$ is the only element of Φ. Hence there is no $X^{\otimes} \in \Phi^{\otimes}$ such that $X^{\otimes} \neq \emptyset$. □

Corollary 16. *If $\Phi \neq \emptyset$ and $\Phi \neq \{\emptyset\}$, then $\Phi^{\otimes} \neq \emptyset$.*

Proof. If $\Phi \neq \emptyset$ and $\Phi \neq \{\emptyset\}$, then there exists $X \in \Phi$ such that $X \neq \emptyset$. Clearly, $X^{\otimes} \neq \emptyset$. On the other hand, $X^{\otimes} \in \Phi^{\otimes}$. □

Now we are ready to prove:

Proposition 9. *For each family of sets Φ there exists a $\chi^{\otimes}(\Phi)$-set.*

Proof. If $\Phi = \emptyset$ or $\Phi = \{\emptyset\}$, then $\emptyset$ is the only $\chi^{\otimes}(\Phi)$-set. If $\Phi \neq \emptyset$ and $\Phi \neq \{\emptyset\}$, then $\Phi^{\otimes} \neq \emptyset$, at least element of $\Phi^{\otimes}$ is a non-empty set, and all the elements of $\Phi^{\otimes}$ are disjoint. Assume that $\emptyset \notin \Phi^{\otimes}$. The existence of $\chi^{\otimes}(\Phi)$-set is now warranted by Corollary 5. Assume that $\emptyset \in \Phi^{\otimes}$. Thus $\emptyset \in \Phi$. Since, by assumption, $\Phi \neq \{\emptyset\}$, we move to $\Psi = \Phi \setminus \{\emptyset\}$. Clearly, the sets in $\Psi^{\otimes}$ are non-empty and disjoint. Thus, by Corollary 5 again, there exists a $\chi^{\otimes}(\Psi)$-set, say, γ. But, obviously, γ is also a $\chi^{\otimes}(\Phi)$-set. □

Proposition 11. *Let $A \in X$ for some $X \in \Phi$. There exists at least one $\chi(\Phi)$-set Z such that $A \in Z$.*

Proof. The family Φ can be displayed as the union of the following families of sets:

$$\Phi_1 = \{X \in \Phi : A \in X\},$$
$$\Phi_2 = \{X \in \Phi : A \notin X\},$$

where $\Phi_1 \cap \Phi_2 = \emptyset$. By Proposition 9, there exists a $\chi^{\otimes}(\Phi_2)$-set, say, γ. Let us define a set δ by the condition:

$$\langle B, Y\rangle \in \delta \textit{ iff } B = A \textit{ and } Y \in \Phi_1$$

Let $\theta = \gamma \cup \delta$. As $\Phi_1 \cap \Phi_2 = \emptyset$, we also have $\gamma \cap \delta = \emptyset$. It is easily seen that $\gamma \cup \delta$ is a $\chi^{\otimes}(\Phi)$-set. Thus $(\gamma \cup \delta)^1$ is a $\chi(\Phi)$-set. Obviously, $A \in (\gamma \cup \delta)^1$. □

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