Logic and Logical Philosophy
Volume 24 (2015), 241–253
DOI: 10.12775/LLP.2015.001

Tomasz Skura
Andrzej Wiśniewski

A SYSTEM FOR PROPER MULTIPLE-CONCLUSION ENTAILMENT

Abstract. The concept of proper multiple-conclusion entailment is introduced. For any sets X, Y of formulas, we say that Y is properly mc-entailed by X iff Y is mc-entailed by X, but no $A \in Y$ is single-conclusion entailed by X. The concept has a natural interpretation in terms of question evocation. A sound and complete axiom system for the propositional case of proper mc-entailment is presented.

Keywords: multiple-conclusion entailment

1. Introduction

1.1. Multiple-conclusion entailment

Multiple-conclusion entailment (mc-entailment for short) is a generalization of the standard, "single-conclusion" entailment (hereafter: sc-entailment). Mc-entailment is a semantic relation between *sets* of well-formed formulas (wffs) of a formal language. Assume that the language in question is supplemented with a semantics rich enough to define some relativized (to a valuation, or a model, etc.) concept of truth for wffs. By and large, a set of wffs X mc-entails a set of wffs Y iff the truth of all the wffs in X warrants the *existence* of at least one true wff in Y. In other words: if all the wffs in X are true, then at least one wff in Y must be true.

Received July 31, 2014. *Revised* January 26, 2015. *Published online* February 4, 2015

1.2. Mc-entailment in CPL

In order to define mc-entailment in exact terms one needs both syntax and semantics of a language in question. Let us consider the case of Classical Propositional Logic (hereafter: CPL). As for syntax, the vocabulary of the language of CPL includes denumerably many propositional variables $p, q, r, s, p_1, \ldots$, and the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$. Wffs are defined in the standard way. We use A, B, C, D, with subscripts if needed, as metalanguage variables for wffs, and X, Y as metalinguistic variables for sets of wffs. As for semantics, a *Boolean valuation* is a function v that assigns a truth value, $\mathbf{1}$ or $\mathbf{0}$, to each propositional variable and is extended to all wffs in the standard manner by using the Boolean functions corresponding to the connectives. Unless otherwise stated, by a valuation we will mean a Boolean valuation. $v(A) = \mathbf{1}$ means "wff A is true under valuation v". Mc-entailment in CPL, $\Vdash$, can now be defined as follows:

Definition 1 (Mc-entailment in CPL). $X \Vdash Y$ iff there is no valuation v such that $v(A) = \mathbf{1}$ for all $A \in X$, and $v(B) = \mathbf{0}$ for every $B \in Y$.

while CPL sc-entailment, $\models$, is defined by:

Definition 2 (CPL sc-entailment). $X \models B$ iff there is no valuation v such that $v(A) = \mathbf{1}$ for all $A \in X$, and $v(B) = \mathbf{0}$.

Observe that when Y is a singleton set, mc-entailment and sc-entailment coincide: $X \Vdash \{B\}$ holds iff $X \models B$ is the case. However, for non-singleton Y's it happens that X mc-entails Y without sc-entailing any wff in Y. For instance, $\{p \vee q\} \Vdash \{p, q\}$ holds, but neither $\{p \vee q\} \models p$ nor $\{p \vee q\} \models q$ is the case. Thus one cannot *define* mc-entailment of a set of wffs as sc-entailment of at least one element of the set. As for CPL, however, mc-entailment is, in a sense, reducible to sc-entailment. We have:[1]

Fact 1. *If X, Y are finite sets of CPL-wffs, then*: $X \Vdash Y$ *iff* $\bigwedge X \models \bigvee Y$.

Yet, since the syntax of CPL does not allow for infinite disjunctions and infinite conjunctions, the reduction does not hold for mc-entailment between infinite sets of wffs. On the other hand, one can show that mc-entailment in CPL is compact, i.e. $X \Vdash Y$ iff $X_1 \Vdash Y_1$ for some finite subsets X_1 of X and Y_1 of Y. Yet, compactness is not an intrinsic property of mc-entailment in general.

[1] We assume that $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$. Note that $\emptyset \nVdash \emptyset$.

Remark 1. One may wonder if mc-entailment between finite sets always coincides with sc-entailment of a disjunction of all the elements of the "conclusion" set from a conjunction of all the elements of the "premise" set. The answer is "No." For instance, take First-order Logic (FoL) and define FoL mc-entailment as follows: X mc-entails Y iff there is no (FoL)model in which all the wffs in X are true and no wff in Y is true.[2] The wff $x = a \vee x \neq a$, where a is an individual constant, is FoL sc-entailed by the empty set. However, the set $\{x = a, x \neq a\}$ is not mc-entailed by the empty set, since there are (FoL)models in which $x = a$ is only satisfied but not true.

Moreover, the analogues of Fact 1 do not hold for some non-classical logics. For example, take a three-valued propositional logic in which disjunction, $\vee$, is understood according to Table 1.[3]

$\vee$	**0**	**i**	**1**
0	**0**	**i**	**1**
i	**i**	**i**	**i**
1	**1**	**1**	**1**

Table 1. McCarthy's disjunction

In such a case $q \vee p$ is not sc-entailed by p because $q \vee p$ can take the value **i** when p takes the (designated) value **1**. On the other hand, the set $\{q, p\}$ is still mc-entailed by the singleton set $\{p\}$. Similarly, $\{p\}$ does not sc-entail $p \vee q$, but mc-entails $\{p, q\}$ when disjunction is construed in a way presented in Table 2.[4]

$\vee$	**0**	**i**	**1**
0	**0**	**i**	**1**
i	**i**	**i**	**i**
1	**1**	**i**	**1**

Table 2. Bochvar's disjunction

[2] By truth in a model we mean here satisfaction by all valuations from the domain of the model.

[3] We borrow the table from [2]. Unlike [2], we use "**1**" for truth, "**0**" for falsity, and "**i**" for the third logical value. As the authors of [2] indicate, the table expresses an idea already present in McCarthy's [8].

[4] This is a table expressing the meaning of disjunction in some of Bochvar's logics; see [3].

1.3. A brief historical note

A syntactic counterpart of mc-entailment is *multiple-conclusion consequence* (mc-consequence for short). It is sometimes claimed that the latter notion originates from Gentzen [6] due to his introduction of sequents of the form $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$. The sign $\Rightarrow$ occurring in a provable sequent can be interpreted as referring to mc-consequence linking the respective sets $\{A_1, \ldots, A_n\}$, and $\{B_1, \ldots, B_k\}$, where the semantic relation between $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_k\}$ is just mc-entailment. Assuming this, a calculus of sequents operating with sequents which have more than one wff on the right side of $\Rightarrow$ is a (single-conclusion) metacalculus for a multiple-conclusion object-level calculus. However, in some cases (Classical Logic included) a sequent $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$ can also be construed as a notational variant of the corresponding wff $A_1 \wedge \cdots \wedge A_n \to B_1 \vee \cdots \vee B_k$, where $\to$ stands for the implication connective. Under this interpretation, Gentzen's calculi of sequents would be variants of the corresponding conventional calculi. Shoesmith and Smiley [10] claim that Gentzen interpreted his calculi of sequents in this latter way. If they are right, it was Carnap who first introduced the concept of mc-entailment (cf. [4]; Carnap uses the term "involution"). Its syntactic counterpart, mc-consequence, was incorporated into the general theory of logical calculi by Dana Scott [9]. Mc-consequence and related concepts (multiple-conclusion calculus, multiple-conclusion rules, etc.) are analysed in detail in the monograph [10].

According to Scott, the following constitute the basic properties of mc-consequence, $\Vdash$ (Scott assumes that X and Y are finite sets of wffs of a language):

(**R**) If $X \cap Y \neq \emptyset$, then $X \Vdash Y$.
(**M**) If $X_1 \Vdash Y_1$, where $X_1 \subseteq X$ and $Y_1 \subseteq Y$, then $X \Vdash Y$.
(**T**) If both $X \Vdash A, Y$ and $X, A \Vdash Y$, then $X \Vdash Y$.

One can easily show that mc-entailment in CPL, $\models\!\!\!\models$, satisfies the above conditions for any finite sets of wffs of the language of CPL and any wff of the language.

In what follows by a *mc-consequence* we will mean a relation between finite sets of wffs that satisfies the conditions (**R**), (**M**), and (**T**).

Remark 2. This definition is very general. Of course, truth/validity-preserving relations are consequence relations. However, falsity/non-validity-preserving relations are consequence relations as well (see e.g. [11, 7]).

1.4. Some interpretational problems

Allowing sets of wffs to constitute conclusions raises interpretational problems. Roughly speaking, Y on the right seems to be a kind of generalized disjunction; let us designate it by $\bigsqcup Y$. There are cases, however, in which this interpretation is problematic.

Following Scott [9, p. 416], we say that a mc-consequence $\Vdash$ is *consistent* iff there is no wff A (of the language in question) such that $\emptyset \Vdash \{A\}$ and $\{A\} \Vdash \emptyset$; $\Vdash$ is *complete* iff for each wff A we have either $\emptyset \Vdash \{A\}$ or $\{A\} \Vdash \emptyset$. As Scott [9] observes, the following holds:

FACT 2. *Every mc-consequence $\Vdash$ is the intersection of all consistent and complete mc-consequences containing $\Vdash$.*

A consistent and complete mc-consequence can be viewed as a *Scott valuation*. For any relation $\Vdash$ of this kind, we define a function $\theta_{\Vdash}$ from the set of wffs to a two-element set $\{\mathbf{t}, \mathbf{f}\}$ as follows:

$$\theta_{\Vdash}(A) = \mathbf{t} \text{ iff } \Vdash \{A\}. \tag{1}$$

(Here and below we write $\Vdash\{A\}$ for $\emptyset\Vdash\{A\}$.) Hence every mc-consequence can also be characterized by the Scott valuations corresponding to the relevant consistent and complete mc-consequences.

On the other hand, every Scott valuation θ determines a mc-consequence $\Vdash_\theta$ by the following condition:

$$\begin{aligned} X \Vdash_\theta Y \text{ iff } \theta(B) = \mathbf{t} \textit{ for some } B \in Y \\ \text{whenever } \theta(A) = \mathbf{t} \textit{ for all } A \in X. \end{aligned}$$

It seems natural to construe $\bigsqcup Y$ as follows. Let θ be a Scott valuation.

$$\theta(\bigsqcup Y) = \mathbf{t} \text{ iff } \theta(B) = \mathbf{t} \text{ for some } B \in Y. \tag{2}$$

However, apart from situations where $\bigsqcup Y$ indeed means "or" (in the object language or in a metalanguage), there are problematic ones. For example, for any Boolean valuation v, define a relation $\Vdash^v$ by:

DEFINITION 3. $X \Vdash^v Y$ iff $v(B) = \mathbf{0}$ for some $B \in Y$ whenever $v(A) = \mathbf{0}$ for all $A \in X$.

It is easy to check that $\Vdash^v$ is a mc-consequence. Also, since v is a function, the relation is consistent and complete. Consider the corresponding Scott valuation $\theta_{\Vdash^v}$. We have: $\theta_{\Vdash^v}(A) = \mathbf{t}$ iff (by (1)) $\Vdash^v\{A\}$

iff (by Definition 3) $v(A) = \mathbf{0}$. Hence, by (2), we get:

$$v(\bigsqcup Y) = \mathbf{0} \text{ iff } v(B) = \mathbf{0} \text{ for some } B \in Y.$$

It seems that $\bigsqcup Y$ is here a conjunction rather than a disjunction

Remark 3. Scott valuations were introduced in [9, p. 416], as valuations corresponding to arbitrary consistent and complete consequence relations. In general, a Scott valuation need not be a truth valuation (that is, a valuation determined by truth tables in which $\mathbf{t}$ means "true"). Note that in the above example $\mathbf{t}$ means "false".

Scott valuations are called models by Gabbay [5]. In [9] the term "$\{\mathbf{t}, \mathbf{f}\}$-valuation" is used.

Anyway, it is worth studying not only the general theory of mc-consequence but also specific types of multiple-conclusion relations having natural motivations.

1.5. Proper mc-entailment

As we have mentioned, mc-entailment generalizes sc-entailment. In particular, we have:

$$X \models A \text{ iff } X \Vdash \{A\}$$

and hence

$$\text{If } X \models A \text{ for some } A \in Y, \text{ then } X \Vdash Y\,. \tag{3}$$

However, the converse of (3) does not hold, that is, it happens that $X \Vdash Y$, but $X \not\models A$ for every $A \in Y$. A simple example has been presented in Section 1.1. Here is another:

$$\{p \wedge q \to r, \neg r\} \Vdash \{\neg p, \neg q\}$$
$$\{p \wedge q \to r, \neg r\} \not\models \neg p$$
$$\{p \wedge q \to r, \neg r\} \not\models \neg q$$

Thus mc-entailment of *non-empty sets* splits into two sub-types: the first, in which a set of wffs is mc-entailed and, at the same time, an element of the set is sc-entailed, and the second, where a set is mc-entailed, but no element of this set is sc-entailed. We will label the second type of mc-entailment of non-empty sets as *proper mc-entailment*, and we will use $\Vert\!\lhd$ as the sign for proper mc-entailment. More precisely, we put:

Definition 4 (Proper mc-entailment). Let $Y \neq \emptyset$. $X \Vert\!\lhd Y$ iff $X \Vdash Y$ and $X \not\models A$ for every $A \in Y$.

Let us note the following.

Remark 4. $\{A\} \not\Vert\!\lhd \{A\}$ for any wff A.

Remark 5. $X \Vert\!\lhd Y$ cannot be expressed as $\bigwedge X \models \bigvee Y$. For example, we have $\{p, q\} \not\Vert\!\lhd \{p, q\}$ but $\{p \wedge q\} \models p \vee q$.

Our aim is to present an axiom system for proper mc-entailment. We remain at the CPL-level. We coin the system PMC.

2. The system PMC

2.1. Terminology and notation

By a *sequent* we mean an expression of the form $X \vdash Y$, where X and Y stand for finite sets of CPL-wffs, and $Y \neq \emptyset$. The antecedent of a sequent can be empty; in such a case we write $\vdash Y$. By a *literal* we mean a propositional variable or the negation of a propositional variable. We say that two literals are *complementary* iff one of them is the negation of the other. A *clause* is a literal or a disjunction of literals. A sequent $X \vdash Y$ is in *normal form* iff every $A \in X$ is a clause and every $B \in Y$ is a conjunction of clauses. By the *rank* of the succedent Y of a sequent in the normal form we mean the number of occurrences of the conjunction connective, $\wedge$, in Y; the rank of Y is designated by $\mathsf{r}(Y)$. We abbreviate "$A_1 \to (A_2 \to \ldots \to (A_n \to B) \ldots)$" as "$\{A_1, \ldots, A_n\} \to B$". The inscription "$A \in \mathsf{CPL}$" means: "$A$ is a thesis of CPL." We characterize finite sets of wffs by listing their elements; curly brackets are thus omitted. As usual, X, A abbreviates $X \cup \{A\}$.

2.2. Axioms

Axioms of PMC are sequents in the normal form falling under the following schema: $\vdash Y$, where Y is of rank 0, $\bigvee Y \in \mathsf{CPL}$, and $B \notin \mathsf{CPL}$ for each $B \in Y$.

Here are simple examples of axioms of PMC:

$$\vdash p, \neg p$$

$$\vdash p \vee \neg q, q \vee \neg p$$

It is easily seen that the following hold:

COROLLARY 1. *If a sequent $\vdash Y$ is an axiom of* PMC, *then no single clause of Y involves complementary literals, but Y involves complementary literals.*

Corollary 2. *If a sequent* $\vdash Y$ *is an axiom of* PMC*, then* $\emptyset \mathrel{\|\!\lhd} Y$*.*

Remark 6. As Remark 4 illustrates, one cannot use $A \vdash A$ as a schema of axioms of PMC. Similarly, $X \vdash X$ is useless. For instance,

$$\{p, \neg p\} \not\mathrel{\|\!\lhd} \{p, \neg p\}.$$

2.3. Rules

Here are the (primary) rules of PMC.

R_1: $$\frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}$$

R_2: $$\frac{X \vdash Y, A}{X \vdash Y, B} \quad \text{where } (A \leftrightarrow B) \in \mathsf{CPL}$$

R_3: $$\frac{X \vdash A \to B_1, \ldots, A \to B_k}{X, A \vdash B_1, \ldots, B_k}$$

The rule R_1 preserves proper mc-entailment (only) from top to bottom, that is, the following holds:

Corollary 3. *If* $X \mathrel{\|\!\lhd} Y, A$ *and* $X \mathrel{\|\!\lhd} Y, B$*, then* $X \mathrel{\|\!\lhd} Y, A \wedge B$*.*

The rules R_2 and R_3, in turn, preserve proper mc-entailment in both directions. To be more precise, we have:

Corollary 4.

1. *Let* $(A \leftrightarrow B) \in \mathsf{CPL}$*. Then* $X \mathrel{\|\!\lhd} Y, A$ *iff* $X \mathrel{\|\!\lhd} Y, B$*.*
2. $X \mathrel{\|\!\lhd} A \to B_1, \ldots, A \to B_k$ *iff* $X, A \mathrel{\|\!\lhd} B_1, \ldots, B_k$*.*

Proof. The case of (1) is obvious.

As for (2), assume that $X \mathrel{\|\!\lhd} A \to B_1, \ldots, A \to B_k$. Hence: (a) $X \mathrel{\|\!\models} A \to B_1, \ldots, A \to B_k$ and (b) $X \not\models A \to B_i$ for $1 \leq i \leq k$. Suppose that $X, A \mathrel{|\!\not\models} B_1, \ldots, B_k$. Hence $X \mathrel{|\!\not\models} A \to B_1, \ldots, A \to B_k$. A contradiction. Suppose that $X, A \models B_i$ for some $1 \leq i \leq k$. Therefore $X \models A \to B_i$. A contradiction again.

The reasoning in the other direction is analogous. □

A *proof* of a sequent $X \vdash Y$ in PMC is a finite labelled tree regulated by the rules of PMC where the leaves are labelled with axioms with axioms and $X \vdash Y$ labels the root. A sequent $X \vdash Y$ is *provable* in PMC iff $X \vdash Y$ has at least one proof in PMC.

Here are examples of proofs:

Example 1. $p \vee \neg p \vdash p, \neg p$

$$\vdash p, \neg p \qquad (\mathrm{Ax})$$
$$\vdash p \vee \neg p \to p, \neg p \qquad (\mathsf{R}_2)$$
$$\vdash p \vee \neg p \to p, p \vee \neg p \to \neg p \qquad (\mathsf{R}_2)$$
$$p \vee \neg p \vdash p, \neg p \qquad (\mathsf{R}_3)$$

Example 2. $p \wedge q \to r, \neg r \vdash \neg p, \neg q$

$$\vdash q \vee r \vee \neg p, p \vee r \vee \neg q \qquad (\mathrm{Ax})$$
$$\vdash \neg q \to r \vee \neg p, p \vee r \vee \neg q \qquad (\mathsf{R}_2)$$
$$\vdash \neg p \vee \neg q \vee r \to r \vee \neg p, p \vee r \vee \neg q \qquad (\mathsf{R}_2)$$
$$\vdash (p \wedge q \to r) \to (\neg r \to \neg p), p \vee r \vee \neg q \qquad (\mathsf{R}_2)$$
$$\vdash (p \wedge q \to r) \to (\neg r \to \neg p), \neg p \to r \vee \neg q \qquad (\mathsf{R}_2)$$
$$\vdash (p \wedge q \to r) \to (\neg r \to \neg p), \neg p \vee \neg q \vee r \to r \vee \neg q \qquad (\mathsf{R}_2)$$
$$\vdash (p \wedge q \to r) \to (\neg r \to \neg p), (p \wedge q \to r) \to (\neg r \to \neg q) \qquad (\mathsf{R}_2)$$
$$p \wedge q \to r \vdash \neg r \to \neg p, \neg r \to \neg q \qquad (\mathsf{R}_3)$$
$$p \wedge q \to r, \neg r \vdash \neg p, \neg q \qquad (\mathsf{R}_3)$$

Example 3. $p \vee q \vdash p, q, r$

$$\vdash \neg q \vee p, \neg p \vee q, \neg p \vee r \quad (\mathrm{Ax}) \qquad\qquad \vdash \neg q \vee p, \neg p \vee q, \neg q \vee r \quad (\mathrm{Ax})$$
$$\vdash \neg q \vee p, \neg p \vee q, (\neg p \vee r) \wedge (\neg q \vee r) \quad (\mathsf{R}_1)$$
$$\vdash p \vee q \to p, \neg p \vee q, (\neg p \vee r) \wedge (\neg q \vee r) \qquad (\mathsf{R}_2)$$
$$\vdash p \vee q \to p, p \vee q \to q, (\neg p \vee r) \wedge (\neg q \vee r) \qquad (\mathsf{R}_2)$$
$$\vdash p \vee q \to p, p \vee q \to q, p \vee q \to r \qquad (\mathsf{R}_2)$$
$$p \vee q \vdash p, q, r \qquad (\mathsf{R}_3)$$

Remark 7. It is natural to ask if the above approach can be extended to non-classical logics. The rule R_1 is obvious. The rule R_2 is a kind of replacement rule. It is R_3 that may be a problem. However, the following holds for many-valued logics. Let v be a valuation in a matrix with a set D of designated values. R_3 is a rule for the logic determined by such a matrix if the following condition is satisfied.

$$v(A \to B) \notin D \text{ iff } v(A) \in D \text{ and } v(B) \notin D.$$

For example, the connective $\supset$ studied by Avron in [1] satisfies this condition.

2.4. Soundness and completeness

The following holds:

Theorem 1 (Soundness of PMC). *If a sequent $X \vdash Y$ is provable in* PMC, *then $X \Vdash\!\!\lhd Y$.*

Proof. By Corollary 2, Corollary 3, and Corollary 4.

In order to prove completeness we need:

Lemma 1. *Let $\vdash Y$ be a sequent in the normal form. If $\emptyset \Vdash\!\!\lhd Y$, then $\vdash Y$ is provable in* PMC.

Proof. We proceed by induction on $\mathsf{r}(Y)$, i.e. the rank of Y.

(1) $\mathsf{r}(Y) = 0$. Assume that $\emptyset \Vdash\!\!\lhd Y$. Then $\vdash Y$ is an axiom of PMC, so the sequent is provable in the calculus.

(2) $\mathsf{r}(Y) > 0$. Assume that $\emptyset \Vdash\!\!\lhd Y$, where $Y = \{A_1, \ldots, A_n\}$. Then there is $1 \leq i \leq n$, say $i = 1$, such that $A_1 = C_1 \wedge \ldots \wedge C_m$ and $m > 1$. At the same time $\not\models A_1$ and thus for some j, where $1 \leq j \leq m$, it holds that $\not\models C_j$. Consider:

$$Y_j = \{C_j, A_2, \ldots, A_n\} \ \ (1 \leqslant j \leqslant m)$$
$$Y'_j = \{\bigwedge\{C_k : k \neq j\}, A_2, \ldots, A_n\}$$

We have $\mathsf{r}(Y_j) < \mathsf{r}(Y)$ and $\mathsf{r}(Y'_j) < \mathsf{r}(Y)$, so by the induction hypothesis:

(a) if $\emptyset \Vdash\!\!\lhd Y_j$, then $\vdash Y_j$ is provable in PMC;
(b) if $\emptyset \Vdash\!\!\lhd Y'_j$, then $\vdash Y'_j$ is provable in PMC.

But when $\emptyset \Vdash\!\!\lhd Y$ holds, we have both $\Vdash\!= Y_j$ and $\Vdash\!= Y'_j$. Yet, it also holds that $\not\models C_j$. Thus $\emptyset \Vdash\!\!\lhd Y_j$ and hence, by (a), $\vdash Y_j$ is provable in PMC.

(*Case 1*) $\not\models \bigwedge\{C_k : k \neq j\}$. Then $\emptyset \Vdash\!\!\lhd Y'_j$, so, by (b), $\vdash Y'_j$ is provable in PMC. Since we have rules R_1 and R_2, and $\vdash Y_j$ is provable as well, it follows that $\vdash Y$ is provable in the calculus.

(*Case 2*) $\models \bigwedge\{C_k : k \neq j\}$. Then A_1 is CPL-equivalent to C_j, so, by R_2, $\vdash Y$ is provable in PMC. □

Lemma 2. *Let $X \vdash Y$, where $X \neq \emptyset$, be a sequent in the normal form. If $X \Vdash\!\!\lhd Y$, then $X \vdash Y$ is provable in* PMC.

PROOF. Let $Y = \{B_1, \ldots, B_k\}$ and $X = \{C_1, \ldots, C_n\}$. Assume that $X \parallel\!\!\lhd Y$. Then, by Corollary 4, we have:

$$\emptyset \parallel\!\!\lhd X \to B_1, \ldots, X \to B_k$$

Since each $X \to B_i$ is CPL-equivalent to a conjunction of clauses, then, by Lemma 1 and rule R_2 the following sequent:

$$\vdash X \to B_1, \ldots, X \to B_k$$

is provable in PMC. We extend the proof of the above sequent by applying rule R_3 n times.[5] As the result we get a proof of the sequent $\{C_1, \ldots, C_n\} \vdash \{B_1, \ldots, B_k\}$, i.e. of $X \vdash Y$. □

THEOREM 2. *Let X, Y be finite sets of* CPL*-wffs. If $X \parallel\!\!\lhd Y$, then the sequent $X \vdash Y$ is provable in* PMC*.*

PROOF. Immediately by Lemma 1 or Lemma 2 if $X \vdash Y$ is in the normal form.

Assume that $X \vdash Y$ is not in the normal form. Suppose that $X \parallel\!\!\lhd Y$. Let $X = \{C_1, \ldots, C_n\}$ and $Y = \{B_1, \ldots, B_k\}$.

By Corollary 4, $X \parallel\!\!\lhd Y$ holds iff

$$\emptyset \parallel\!\!\lhd X \to B_1, \ldots, X \to B_k \tag{4}$$

is the case.

Each of $X \to B_i$ is CPL-equivalent to a conjunction D_i of clauses. Clearly, (4) holds iff the following is the case:

$$\emptyset \parallel\!\!\lhd D_1, \ldots, D_k$$

Observe that the corresponding sequent:

$$\vdash D_1, \ldots, D_k \tag{5}$$

is in the normal form. Therefore, by the initial assumption and Lemma 1, the sequent (5) is provable in PMC. By applying rule R_2 k times one can extend a proof of (5) into a proof of the sequent:

$$\vdash X \to B_1, \ldots, X \to B_k \tag{6}$$

Then, by applying rule R_3 n times one can extend a proof of (6) into a proof of the sequent $X \vdash Y$. □

[5] Recall that "$X \to B_i$" abbreviates "$C_1 \to (C_2 \to \ldots \to (C_n \to B_i)\ldots)$".

What about proper mc-entailment between infinite sets of CPL-wffs? The following holds:

COROLLARY 5. *If* $X \Vert\!\lhd Y$, *then* $X_1 \Vert\!\lhd Y_1$ *for some finite sets* X_1, Y_1 *such that* $X_1 \subseteq X$ *and* $Y_1 \subseteq Y$.

Hence the following is true:

THEOREM 3 (Weak completeness of PMC). *If* $X \Vert\!\lhd Y$, *then there exists a sequent* $X_1 \vdash Y_1$ *such that* $X_1 \subseteq X$ *as well as* $Y_1 \subseteq Y$, *and* $X_1 \vdash Y_1$ *is provable in* PMC.

3. Proper mc-entailment and question evocation

As we mentioned in Section 1.4, allowing sets of wffs to constitute conclusions raises interpretational problems. However, the case in which a set of wffs is properly mc-entailed seems less problematic. It is quite natural to think of a properly mc-entailed set of wffs as of the set of *direct answers to a question*, where a direct answer is a possible answer that provides neither less nor more information than is required by the question. Assuming this, the first clause of the definition of proper mc-entailment amounts to: if all the wffs in X are true, then the question whose set of direct answers is Y must be sound, i.e. at least one direct answer to the question must be true. The second clause, in turn, says the following: no direct answer to the question is (sc-)entailed by X. Thus the truth of all the wffs in X warrants the *existence* of a true direct answer but does not determine *which* direct answer is true. In other words: the question is sound relative to X, but it expresses a problem which is open with respect to X. A reader familiar with Inferential Erotetic Logic (IEL for short) immediately notices that $X \Vert\!\lhd Y$ holds iff X *evokes* a question whose set of direct answers is Y.[6] The concept of question evocation, however, plays a crucial role in IEL. In particular, *validity* of inferences leading from declaratives to questions is defined in terms of evocation. PMC can thus be interpreted as an axiom system for question evocation. Let us add: the first known system of this kind.

Funding. This work was supported by funds of the National Science Centre, Poland (DEC-2012/04/A/HS1/00715 to A.W.).

[6] For IEL and question evocation see, e.g., [12].

References

[1] Avron, A., "Natural 3-valued logics – characterization and proof theory", *Journal of Symbolic Logic*, 56 (1991): 276–294. DOI: 10.2307/2274919

[2] Avron, A., and A. Zamansky, "Completeness and axiomatizability in many-valued logic", pages 227–304 in volume 16 of *Handbook of Philosophical Logic*, Springer, Dordrecht, 2011.

[3] Bolc, L., and P. Borowik, *Many-Valued Logics 1. Theoretical Foundations*, Springer, Berlin/Heilderberg/New York, 1992.

[4] Carnap, R., *Formalization of Logic*, Harvard University Press, Cambridge, MA, 1943.

[5] Gabbay, D. M., *Semantical Investigations in Heyting's Intuitionistic Logic*, Reidel, Dordrecht, 1981.

[6] Gentzen, G., "Investigation into logical deduction", pages 68–131 in *The Collected Papers of Gerhard Gentzen*, M. E. Szabo (ed.), North-Holland, 1969.

[7] Kracht, M., "Judgement and consequence relations", *Journal of Applied Non-Classical Logics*, 20 (2010): 423–435. DOI: 10.3166/jancl.20.423-435

[8] McCarthy, J., "A basis for a mathematical theory of computation", pages 33–70 in *Computer Programming and Formal Systems*, P. Braffort and D. Hirshberg (eds.), North-Holland, Amsterdam, 1963.

[9] Scott, D., "Completeness and axiomatizability in many-valued logic", pages 411–435 in volume 25 of *Proceedings of Symposia in Pure Mathematics*, American Mathematican Society, Providence, Rhode Island, 1974.

[10] Shoesmith, D. J., and T. J. Smiley *Multiple-conclusion Logic*, Cambridge University Press, Cambridge, 1978.

[11] Skura, T., "A refutation theory", *Logica Universalis*, 3 (2009): 293–302. DOI: 10.1007/s11787-009-0009-y

[12] Wiśniewski, A., *Questions, Inferences, and Scenarios*, vol. 46 of *Studies in Logic*, College Publications, London, 2013.

Tomasz Skura
Institute of Philosophy
University of Zielona Góra
Zielona Góra, Poland
`T.Skura@ifil.uz.zgora.pl`

Andrzej Wiśniewski
Department of Logic and Cognitive Science
Institute of Psychology
Adam Mickiewicz University
Poznań, Poland
`Andrzej.Wisniewski@amu.edu.pl`