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A Right-sided Socratic Calculus for Classical Logic

Research Report[1]

0. Aims

The aim of this paper is to present a certain erotetic calculus for first-order logic. This calculus is an alternative to the calculus $\mathbf{E}^{\mathbf{PQ}}$ presented in Wiśniewski and Shangin (2006). The most important feature of the current calculus is that it operates on right-sided sequents only. The right-sided approach is natural when modal logics are analyzed in an erotetic setting (see Leszczyńska 2004, 2006). The propositional part of our calculus (never presented in a written form, but communicated during some meetings) constitutes the background for the appropriate modal Socratic calculi.

1. Syntax and Semantics. Terminology and Notation

We use the language $\boldsymbol{L}$ of Pure Calculus of Quantifiers, described in Wiśniewski and Shangin (2006), as the point of departure. The vocabulary of $\boldsymbol{L}$ includes parameters, but these do not occur in the so-called pure sentences, by means of which laws of logic can be expressed. We introduce an "erotetic" language, $\boldsymbol{L}^{**}$, which resembles the erotetic language $\boldsymbol{L}^{*}$ for $\mathbf{E}^{\mathbf{PQ}}$; both languages have *declarative well-formed formulas* (d-wffs) and *questions* as meaningful expressions, and are built according to a common pattern. The only substantial difference lies in the fact that we now consider right-sided sequents (and only them) as 'bricks', out of which both d-wffs and questions of $\boldsymbol{L}^{**}$ are constructed. A *right-sided sequent* is an expression of the form:

(1) $\vdash S$

where S is a non-empty finite sequence of sentences (*i.e.* closed well-formed formulas) of $\boldsymbol{L}$. Note that a right-sided sequent *is not* an expression of $\boldsymbol{L}$. In practice, we will be writing $\vdash A_1, ..., A_n$ instead of $\vdash < A_1, ..., A_n >$. A sequent is *pure* if it involves only parameter-free sentences. The remaining syntactic and semantic concepts are defined accordingly; we use the terminology and notation of (Wiśniewski and Shangin, 2006).

A sequent of the form $\vdash S$ is valid iff there is no model of $\boldsymbol{L}$ in which all the elements of S are false. Thus $\vdash A_1, ..., A_n$ is valid iff $A_1 \vee (A_2 \vee ... \vee (A_{n-1} \vee A_n)...))$ is valid.

2. The Calculus $\mathbf{E}^{\mathbf{RPQ}}$

We shall coin our new calculus with the name $\mathbf{E}^{\mathbf{RPQ}}$. However, before we present it, let us remind some notational conventions used in the presentation of "old" erotetic calculus for Pure Calculus of Quantifiers, *i.e.* $\mathbf{E}^{\mathbf{PQ}}$.

α	α_1	α_2		β	β_1	β_2	$\beta_1{}^*$
$A \wedge B$	A	B		$\neg(A \wedge B)$	$\neg A$	$\neg B$	A
$\neg(A \vee B)$	$\neg A$	$\neg B$		$A \vee B$	A	B	$\neg A$
$\neg(A \rightarrow B)$	A	$\neg B$		$A \rightarrow B$	$\neg A$	B	A

Table 1.

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κ	κ^*
$\neg\neg A$	A
$\neg\exists x_i A$	$\forall x_i \neg A$
$\neg\forall x_i A$	$\exists x_i \neg A$
$\forall x_i A$, *provided that* x_i *is not free in* A	A
$\exists x_i A$, *provided that* x_i *is not free in* A	A

Table 2.

$\mathbf{E}^{\mathbf{PQ}}$ is a calculus of *questions*, and the inferential rules of the calculus transform a question into a question. A question of the language $\boldsymbol{L}^*$ is an expression of the form:

(2) $\quad ?\,(S_1 \vdash A_1,\ \ldots\ ,\ S_n \vdash A_n)$

where $n \geq 1$, $S_1, \ldots, S_n$ are finite (possibly empty) sequences of sentences of $\boldsymbol{L}$, and $A_1, \ldots, A_n$ are sentences of $\boldsymbol{L}$. Thus a question of $\boldsymbol{L}^*$ is based on a finite sequence of single-conclusioned sequents (these are expressions of $\boldsymbol{L}^*$, not of **L** !). A sequent occurring is a question is called a *constituent* of the question. An intuitive reading of a question of the form (2) is: "*Is it the case that*: $S_1 \vdash A_n$ *is FOL-valid and … and* $S_n \vdash A_n$ *is FOL-valid*?", where the concept of First-Order Logic (FOL) validity of a sequent is understood in the standard manner. Thus a question asks about *joint validity* of sequents involved in it. We use the semicolon ";" as the concatenation-sign for sequences of sequents. A metalinguistic expression of the form:

(3) $\quad \Phi;\ S \vdash A$

refers to a sequence of sequents which is the concatenation of a finite (possibly empty) sequence of sequents Φ and the one-term sequence $<S \vdash A>$. Similarly, an expression of the form:

(4) $\quad \Phi;\ S \vdash A;\ \Psi$

represents the concatenation of $\Phi;\ S \vdash A$ and a finite (possibly empty) sequence of sequents Ψ. The sign ' is the concatenation-sign for sequences of sentences of $\boldsymbol{L}$. By $S\,'A$ we mean the concatenation of a sequence S of sentences of $\boldsymbol{L}$ and the one-term sequence $<A>$, where A is a sentence of $\boldsymbol{L}$. The reading of a metalinguistic inscription of the form $S\,'A\,'T$ is analogous to that of (4). Both S and T can be empty.

We remind the (primary) inferential rules of $\mathbf{E}^{\mathbf{PQ}}$:

$\mathbf{L}_\alpha$: $\quad \dfrac{?\,(\Phi;\ S\,'\alpha\,'T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\alpha_1\,'\alpha_2\,'T \vdash C;\ \Psi)}$ $\qquad$ $\mathbf{R}_\alpha$: $\quad \dfrac{?\,(\Phi;\ S \vdash \alpha;\ \Psi)}{?\,(\Phi;\ S \vdash \alpha_1;\ S \vdash \alpha_2;\ \Psi)}$

$\mathbf{L}_\beta$: $\quad \dfrac{?\,(\Phi;\ S\,'\beta\,'T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\beta_1\,'T \vdash C;\ S\,'\beta_2\,'T \vdash C;\ \Psi)}$ $\qquad$ $\mathbf{R}_\beta$: $\quad \dfrac{?\,(\Phi;\ S \vdash \beta;\ \Psi)}{?\,(\Phi;\ S\,'\beta_1{}^* \vdash \beta_2;\ \Psi)}$

$\mathbf{L}_\kappa$: $\quad \dfrac{?\,(\Phi;\ S\,'\kappa\,'T \vdash C;\ \Psi)}{?\,(\Phi;\ S\,'\kappa^*\,'T \vdash C;\ \Psi)}$ $\qquad$ $\mathbf{R}_\kappa$: $\quad \dfrac{?\,(\Phi;\ S \vdash \kappa;\ \Psi)}{?\,(\Phi;\ S \vdash \kappa^*;\ \Psi)}$

L$_\forall$:

$$\frac{?\,(\Phi;\, S\,'\forall x_i A\,'T \vdash B;\, \Psi)}{?\,(\Phi;\, S\,'\forall x_i A\,'A(x_i/\tau)\,'T \vdash B;\, \Psi)}$$

provided that x_i is free in A;
τ is any parameter

R$_\forall$:

$$\frac{?\,(\Phi;\, S \vdash \forall x_i A;\, \Psi)}{?\,(\Phi;\, S \vdash A(x_i/\tau);\, \Psi)}$$

provided that x_i is free in A,
and τ is a parameter which does
not occur in $S \vdash \forall x_i A$

L$_\exists$:

$$\frac{?\,(\Phi;\, S\,'\exists x_i A\,'T \vdash B;\, \Psi)}{?\,(\Phi;\, S\,'A(x_i/\tau)\,'T \vdash B;\, \Psi)}$$

provided that x_i is free in A,
and τ is a parameter which
does not occur in $S\,'\exists x_i A\,'T \vdash B$

R$_\exists$:

$$\frac{?\,(\Phi;\, S \vdash \exists x_i A;\, \Psi)}{?\,(\Phi;\, S\,'\forall x_i \neg A \vdash A(x_i/\tau);\, \Psi)}$$

provided that x_i is free in A;
τ is any parameter

Rules of $\mathbf{E}^{\mathbf{RPQ}}$ will be presented in a format similar to that of rules of $\mathbf{E}^{\mathbf{PQ}}$.

In order to characterize inferential rules of $\mathbf{E}^{\mathbf{RPQ}}$, however, we also have to define the syntactical relation oqe (after "obvious quantificational equivalence") among sentences of ***L***. For brevity, we adopt the following notational convention: if ∇ is $\forall$, then Δ is $\exists$; if ∇ is $\exists$ then Δ is $\forall$.

Definition 1.

(i) $\nabla x_i A$ oqe $\neg \Delta x_i \neg A$,
(ii) $\neg \Delta x_i \neg A$ oqe $\nabla x_i A$,
(iii) $\neg \nabla x_i A$ oqe $\Delta x_i \neg A$,
(iv) $\Delta x_i \neg A$ oqe $\neg \nabla x_i A$,
(v) *nothing else stands in the relation* oqe.

Here is the complete list of *primary inferential rules* of $\mathbf{E}^{\mathbf{RPQ}}$:

R$_\alpha$

$$\frac{?\,(\Phi;\, \vdash S\,'\alpha\,'T;\, \Psi)}{?\,(\Phi;\, \vdash S\,'\alpha_1\,'T;\, \vdash S\,'\alpha_2\,'T;\, \Psi)}$$

R$_\beta$

$$\frac{?\,(\Phi;\, \vdash S\,'\beta\,'T;\, \Psi)}{?\,(\Phi;\, \vdash S\,'\beta_1\,'\beta_2\,'T;\, \Psi)}$$

R$_{\neg\neg}$

$$\frac{?\,(\Phi;\, \vdash S\,'\neg\neg A\,'T;\, \Psi)}{?\,(\Phi;\, \vdash S\,'A\,'T;\, \Psi)}$$

R$_\exists$

$$\frac{?\,(\Phi;\, \vdash S\,'\exists x_i A\,'T;\, \Psi)}{?\,(\Phi;\, \vdash S\,'\exists x_i A\,'A(x_i/\tau)\,'T;\, \Psi)}$$

provided that x_i is free in A; τ is any parameter

R$_\forall$

$$\frac{?\,(\Phi;\, \vdash S\,'\forall x_i A\,'T;\, \Psi)}{?\,(\Phi;\, \vdash S\,'A(x_i/\tau)\,'T;\, \Psi)}$$

provided that x_i is free in A, and τ is a parameter which does not occur in $\vdash S\,'\forall x_i A\,'T$.

R$_{\forall *}$

$$\frac{?\,(\Phi;\, \vdash S\,'\forall x_i A\,'T;\, \Psi)}{?\,(\Phi;\, \vdash S\,'A\,'T;\, \Psi)}$$

provided that x_i is not free in A

$$\mathbf{R}_{\exists *} \quad \frac{?\,(\Phi;\ \vdash S\,'\exists x_i A\,'T;\ \Psi)}{?\,(\Phi;\ \vdash S\ 'A\ 'T;\ \Psi)}$$
provided that x_i is not free in A

$$\mathbf{R}_{\underline{oqe}} \quad \frac{?\,(\Phi;\ \vdash S\ 'A\ 'T;\ \Psi)}{?\,(\Phi;\ \vdash S\ 'B\ 'T;\ \Psi}$$
provided that B <u>oqe</u> *A*.

Lemma 1: *Primary rules of* $\mathbf{E}^{\mathbf{RPQ}}$ *preserve the transmission of joint validity of sequents in both directions, that is, if Q* results from Q by an application of a primary rule of* $\mathbf{E}^{\mathbf{RPQ}}$*, then each constituent (sequent) of Q* is valid if and only if each constituent (sequent) of Q is valid.*

P r o o f: By cases. ■

Since we will not consider derived rules here, in what follows by rules of $\mathbf{E}^{\mathbf{PRQ}}$ we will mean primary rules of $\mathbf{E}^{\mathbf{PRQ}}$. Socratic transformations are defined in the standard way.

Definition 2: *A sequence of questions* $<Q_1, Q_2, \ldots>$ *of* $\boldsymbol{L}^{**}$ *is a* Socratic transformation *of a question Q* via $\mathbf{E}^{\mathbf{RPQ}}$ *iff* $Q_1 = Q$*, and* Q_{i+1} *results from* Q_i $(i \geq 1)$ *by an application of a rule of* $\mathbf{E}^{\mathbf{RPQ}}$.

We say that a finite Socratic transformation *leads to* a question Q_i iff Q_i is the last term of the transformation.

Recall that a pure sequent is a sequent in which only pure sentences (*i.e.* sentences which do not involve any parameters) occur. The concept of a Socratic proof is defined by:

Definition 3: *Let* $\vdash A$ *be a pure sequent. A* Socratic proof *of* $\vdash A$ *in* $\mathbf{E}^{\mathbf{RPQ}}$ *is a finite Socratic transformation of* ? ($\vdash A$) *via* $\mathbf{E}^{\mathbf{RPQ}}$ *such that for each constituent* ϕ *of the last question of the transformation:*[2]

(a) ϕ *is of the form* $\vdash T\,'B\,'U\,'\neg B\,'W$, *or*
(b) ϕ *is of the form* $\vdash T\,'\neg B\,'U\,'B\,'W$.

If sequent $\vdash A$ *has a Socratic proof in* $\mathbf{E}^{\mathbf{PQ}}$*, we say that* $\vdash A$ *is* provable *in* $\mathbf{E}^{\mathbf{PQ}}$*. Moreover, we say that the sentence A is* provable *in* $\mathbf{E}^{\mathbf{RPQ}}$*. A constituent of the form* (a) *or* (b) *is called* successful.

Let us stress that, according to Definition 2, each Socratic proof in $\mathbf{E}^{\mathbf{RPQ}}$ must begin with a question based on a pure sequent. An analogous restriction is imposed in $\mathbf{E}^{\mathbf{PQ}}$.

Corollary 1. *Any sequent of the form* (a) *or* (b) *specified in Definition 2 is valid.*

Theorem 1 (soundness of $\mathbf{E}^{\mathbf{PRQ}}$) *If* $\vdash A$ *is provable in* $\mathbf{E}^{\mathbf{RPQ}}$*, then A is valid.*

P r o o f: Each constituent of the last question of a Socratic proof of $\vdash A$ is valid. Hence, by Lemma 1, the sequent $\vdash A$ is valid, and thus the sentence *A* is valid. ■

3. A Short Comparison of $\mathbf{E}^{\mathbf{PQ}}$ and $\mathbf{E}^{\mathbf{PRQ}}$

Observe that the propositional rules of both calculi are eliminative: an application of a rule amounts to the elimination of a binary connective or a double negation. Consecutive applications of quantificational rules of $\mathbf{E}^{\mathbf{PQ}}$ may result in the elimination of all quantificational

[2] Since we do not have structural rules, both (a) and (b) are needed.

formulas with the exception of formulas of the form $\forall x_i\, A$, where x_i is free in A. $\mathbf{E}^{\mathbf{RPQ}}$ has a similar property with regard to formulas of the form $\exists x_i\, A$ and $\neg\forall x_i\, A$ (with x_i free in A).

Another feature of $\mathbf{E}^{\mathbf{RPQ}}$ is that it allows for repeating questions (although only in a somehow stupid way; a Socratic transformation of the kind $<Q, Q', Q>$ is permitted due to the presence of rule $\boldsymbol{R}_{\underline{\mathbf{oqe}}}$)[3]. On the other hand, repetitions of questions never happen in Socratic transformations via $\mathbf{E}^{\mathbf{PQ}}$.

As far as $\mathbf{E}^{\mathbf{PQ}}$ is concerned, the complexity of a "new" wff is not greater than the complexity of the "old" formula. This does not hold in $\mathbf{E}^{\mathbf{RPQ}}$. Again, rule $\boldsymbol{R}_{\underline{\mathbf{oqe}}}$ is the reason.

However, it seems that rule $\boldsymbol{R}_{\underline{\mathbf{oqe}}}$ is natural in a "Socratic" erotetic setting. The underlying idea is: *use that one of semantically equivalent quantificational formulas which is convenient in a given context*.

4. Completeness of $\mathbf{E}^{\mathbf{RPQ}}$

In the completeness proof of $\mathbf{E}^{\mathbf{RPQ}}$ we shall use an indirect method (although a direct proof is also possible). The general idea of our proof is the following. Since $\mathbf{E}^{\mathbf{PQ}}$ is complete (see Wiśniewski and Shangin 2006), each valid sequent of the form $\vdash A$ is provable in $\mathbf{E}^{\mathbf{PQ}}$. We will show that for each (Socratic) proof of $\vdash A$ *in* $\mathbf{E}^{\mathbf{PQ}}$ there exists a "parallel" (Socratic) proof of $\vdash A$ *in* $\mathbf{E}^{\mathbf{RPQ}}$. We proceed in a "constructive" way, showing how a proof in $\mathbf{E}^{\mathbf{RPQ}}$ can be extracted from a given proof in $\mathbf{E}^{\mathbf{PQ}}$.

First we introduce some auxiliary concepts:

Definition 4 (duals): (i) *If A is of the form $\neg B$, then $A^{\#} = B$*; (ii) *if A is not of the form $\neg B$, then $A^{\#} = \neg A$.*

Definition 5 (dual sequences of wffs): If $S = <B_1, ..., B_n>$, *then:* (a) $S^{\#} = S$ *if* $n = 0$; (ii) $S^{\#} = <B_1{}^{\#}, ..., B_n{}^{\#}>$ *if* $n > 0$.

Definition 6 (dual sequents): $(S \vdash A)^{\#} = \vdash S^{\#}\, 'A$

Definition 6 (dual questions): $(?\,(S_1 \vdash A_1; ...; S_n \vdash A_n))^{\#} = ?\,(\vdash S_1{}^{\#}\, 'A_1; ...; \vdash S_n{}^{\#}\, 'A_n)$

We shall prove:

Lemma 2: *Let* $\mathbf{s} = <Q_1, ..., Q_n>$ *be a Socratic proof of* $\vdash A$ in $\mathbf{E}^{\mathbf{PQ}}$. *Then for each index i such that* $1 < i \leq n$ *the following holds*: *either* $Q_i{}^{\#} = Q_{i-1}{}^{\#}$, *or* $Q_i{}^{\#}$ *results from* $Q_{i-1}{}^{\#}$ *by a rule of* $\mathbf{E}^{\mathbf{RPQ}}$, *or* $Q_i{}^{\#}$ *does not result from* $Q_{i-1}{}^{\#}$ *by a rule of* $\mathbf{E}^{\mathbf{RPQ}}$, *but there exists a finite Socratic transformation of* $Q_{i-1}{}^{\#}$ *via* $\mathbf{E}^{\mathbf{RPQ}}$ *that leads to* $Q_i{}^{\#}$.

P r o o f: Since $\mathbf{s}$ is a Socratic proof in $\mathbf{E}^{\mathbf{PQ}}$, Q_i ($1 < i < n$) results from Q_{i-1} by applying a rule of $\mathbf{E}^{\mathbf{PQ}}$. We have to consider the following cases:

I: Propositional cases

<u>*Case 1.1*</u>: Rule $\mathbf{L}_{\neg\neg}$ was applied to Q_{i-1} with respect to a wff, say, $\neg\neg A$. Now $Q_{i-1}{}^{\#} = ?\,(\Phi;\ \vdash S\#\ '\ (\neg\neg A)^{\#}\ 'T^{\#}\ 'B)$ and $Q_i{}^{\#} = ?\,(\Phi;\ S^{\#}\ '(A)^{\#}\ 'T^{\#}\ 'B)$. Let us observe that $(\neg\neg A)^{\#} = (A)^{\#} = \neg A$ if A is not of the form $\neg D$; otherwise $(\neg\neg A)^{\#} = \neg\neg D$ and $(A)^{\#} = D$. Hence either $Q_i{}^{\#} = Q_{i-1}{}^{\#}$, or $Q_i{}^{\#}$ results from $Q_{i-1}{}^{\#}$ by $\boldsymbol{R}_{\neg\neg}$.

[3] An easy remedy for this is to require the elements of a Socratic transformation to be syntactically different. This would complicate the metatheory, however.

Case 1.2: Rule $\mathbf{L}_\beta$ was applied to $Q_{i\text{-}1}$ with respect to a given β-wff. Now $Q_{i\text{-}1}^{\#} = ?\,(\Phi;\ \vdash S^{\#}\,'\beta^{\#}\,'T^{\#}\,'B;\ \Psi)$ and $Q_i^{\#} = ?\,(\Phi;\ \vdash S^{\#}\,'\beta_1^{\#}\ \,'T^{\#}\,'B;\ \vdash S^{\#}\,'\beta_2^{\#}\ \,'T^{\#}\,'B;\ \Psi)$.

Assume that $\beta = A \vee C$. We have $\beta^{\#} = \neg(A \vee C)$, $\beta_1^{\#} = (A)^{\#}$, and $\beta_2^{\#} = (C)^{\#}$. If neither A nor C is of the form $\neg D$, then $(A)^{\#} = \neg A$ and $(B)^{\#} = \neg B$. Hence $Q_i^{\#}$ results from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_{\neg\vee}$. Now suppose that A is of the form $\neg D$, and that C is of the form $\neg E$. In this case $(A)^{\#} = D$ and $(C)^{\#} = E$. However, the following Socratic transformation[4] of $Q_{i\text{-}1}^{\#}$ via $\mathbf{E}^{\mathbf{RPQ}}$ leads to $Q_i^{\#}$:

$\mathbf{T}(\mathbf{L}_\vee / \boldsymbol{R}_{\neg\vee}, \boldsymbol{R}_{\neg\neg}, \boldsymbol{R}_{\neg\neg})$:

$?\,(\Phi;\ \vdash S^{\#}\,'\neg(\neg D \vee \neg E)\,'T^{\#}\,'B;\ \Psi)$	$\boldsymbol{R}_{\neg\vee}$
$?\,(\Phi;\ \vdash S^{\#}\,'\neg\neg D\,'T^{\#}\,'B; \vdash S_{\#}\,'\neg\neg E\,'T^{\#}\,'B;\ \Psi)$	$\boldsymbol{R}_{\neg\neg}$
$?\,(\Phi;\ \vdash S^{\#}\,'D\,'T^{\#}\,B; \vdash S^{\#}\,'\neg\neg E\ \,'T^{\#}\,'B;\ \Psi)$	$\boldsymbol{R}_{\neg\neg}$
$?\,(\Phi;\ \vdash S^{\#}\,'D\,'T^{\#}\,'B; \vdash S^{\#}\,'E\,'T^{\#}\,'B;\ \Psi)$	

It is obvious that if only one of $\{A, C\}$ begins with ¬, then there exists a Socratic transformation of $Q_{i\text{-}1}^{\#}$ via $\mathbf{E}^{\mathbf{RPQ}}$ which leads to $Q_i^{\#}$; this transformation differs from the previous one in applying rule $\boldsymbol{R}_{\neg\neg}$ only once. These transformations (recall that we have two possibilities here) can be designated by $\mathbf{T_1}(\mathbf{L}_\vee / \boldsymbol{R}_{\neg\vee}, \boldsymbol{R}_{\neg\neg})$ and $\mathbf{T_2}(\mathbf{L}_\vee / \boldsymbol{R}_{\neg\vee}, \boldsymbol{R}_{\neg\neg})$, respectively. We leave their description to the reader.

Assume that $\beta = A \rightarrow C$. Thus $\beta^{\#} = \neg(A \rightarrow C)$ and $\beta_1^{\#} = A$. Suppose that C is not of the form $\neg D$. Hence $\beta_2^{\#} = \neg C$. Therefore $Q_i^{\#}$ results from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_{\neg\rightarrow}$. Now suppose that C is of the form $\neg D$. Thus $\beta_2^{\#} = D$. However, the following is a transformation of $Q_{i\text{-}1}^{\#}$ via $\mathbf{E}^{\mathbf{RPQ}}$ which leads to $Q_i^{\#}$:

$\mathbf{T}(\mathbf{L}_\rightarrow / \boldsymbol{R}_{\neg\rightarrow}, \boldsymbol{R}_{\neg\neg})$:

$?\,(\Phi;\ \vdash S^{\#}\,'\neg(A \rightarrow \neg D)\,'T^{\#}\,'B;\ \Psi)$	$\boldsymbol{R}_{\neg\rightarrow}$
$?\,(\Phi;\ \vdash S^{\#}\,'A\,'T^{\#}\,'B;\ \vdash S^{\#}\,'\neg\neg D\,'T^{\#}\,'B;\ \Psi)$	$\boldsymbol{R}_{\neg\neg}$
$?\,(\Phi;\ \vdash S^{\#}\,'D\,'T^{\#}\,'B;\ \vdash S^{\#}\,'D\,'T^{\#}\,'B;\ \Psi)$	

Assume that $\beta = \neg(A \wedge B)$. Hence $\beta^{\#} = A \wedge B$, $\beta_1^{\#} = A$, and $\beta_2^{\#} = B$. Therefore $Q_i^{\#}$ results from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_{\wedge}$.

Case 1.3: Rule $\mathbf{L}_\alpha$ was applied to $Q_{i\text{-}1}$ with respect to a given α-wff. We have $Q_{i\text{-}1}^{\#} = ?\,(\Phi;\ \vdash S^{\#}\,'\alpha^{\#}\,'T^{\#}\,'B;\ \Psi)$ and $Q_i^{\#} = ?\,(\Phi;\ \vdash S^{\#}\,'\alpha_1^{\#}\,'\alpha_2^{\#}\,'T^{\#}\,'B;\ \Psi)$.

Assume that $\alpha = A \wedge C$. Hence $\alpha^{\#} = \neg(A \wedge C)$. Suppose that neither A nor C begins with ¬. Hence $\alpha_1^{\#} = \neg A$ and $\alpha_2^{\#} = \neg C$. Thus $Q_i^{\#}$ results from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_{\neg\wedge}$. Now suppose that $A = \neg D$, but C is not of the form $\neg E$. It follows that $\alpha_1^{\#} = D$ and $\alpha_2^{\#} = \neg C$. Let us now consider the following Socratic transformation of $Q_{i\text{-}1}^{\#}$ via $\mathbf{E}^{\mathbf{RPQ}}$:

$\mathbf{T_1}(\mathbf{L}_\wedge / \boldsymbol{R}_{\neg\wedge}, \boldsymbol{R}_{\neg\neg})$:

$?\,(\Phi;\ \vdash S^{\#}\,'\neg(\neg D \wedge C)\,'T^{\#}\,'B;\ \Psi)$	$\boldsymbol{R}_{\neg\wedge}$
$?\,(\Phi;\ \vdash S^{\#}\,'\neg\neg D\,'\neg C\,'T^{\#}\,'B;\ \Psi)$	$\boldsymbol{R}_{\neg\neg}$
$?\,(\Phi;\ \vdash S^{\#}\,'D\,'\neg C\,'T^{\#}\,'B;\ \Psi)$	

The above transformation leads to $Q_i^{\#}$. If A is not of the form $\neg D$, but $C = \neg E$, we have the following transformation via $\mathbf{E}^{\mathbf{RPQ}}$ with the desired property

[4] For transparency, we highlight the sentence of $\boldsymbol{L}$ acted upon.

$\mathbf{T_2}(\mathbf{L}_{\wedge} / \boldsymbol{R}_{\neg\wedge}, \boldsymbol{R}_{\neg\neg})$:

$? (\Phi; \vdash S^{\#}\text{'}\neg(B \wedge \neg E)\text{'}T^{\#}\text{'}B; \Psi)$ $\quad \boldsymbol{R}_{\neg\wedge}$

$? (\Phi; \vdash S^{\#}\text{'}\neg B\text{'}\neg\neg E\text{'}T^{\#}\text{'}B; \Psi)$ $\quad \boldsymbol{R}_{\neg\neg}$

$? (\Phi; \vdash S^{\#}\text{'}\neg B\text{'}E\text{'}T^{\#}\text{'}B; \Psi)$

If $B = \neg D$ and $C = \neg E$, then $\alpha_1^{\#} = D$ and $\alpha_2^{\#} = E$. In this case the following transformation via $\mathbf{E}^{\mathbf{RPQ}}$ leads to $Q_i^{\#}$:

$\mathbf{T}(\mathbf{L}_{\wedge} / \boldsymbol{R}_{\neg\wedge}, \boldsymbol{R}_{\neg\neg}, \boldsymbol{R}_{\neg\neg})$

$? (\Phi; \vdash S^{\#}\text{'}\neg(\neg D \wedge \neg E)\text{'}T^{\#}\text{'}B; \Psi)$ $\quad \boldsymbol{R}_{\neg\wedge}$

$? (\Phi; \vdash S^{\#}\text{'}\neg\neg D\text{'}\neg\neg E\text{'}T^{\#}\text{'}B; \Psi)$ $\quad \boldsymbol{R}_{\neg\neg}$

$? (\Phi; \vdash S^{\#}\text{'}D\text{'}\neg\neg E\text{'}T^{\#}\text{'}B; \Psi)$ $\quad \boldsymbol{R}_{\neg\neg}$

$? (\Phi; \vdash S^{\#}\text{'}D\text{'}E\ T^{\#}\text{'}B; \Psi)$

Assume that $\alpha = \neg(A \to C)$. Now $\alpha^{\#} = A \to C$ and $\alpha_2^{\#} = C$ (since $\alpha_2 = \neg C$). Suppose that A is not of the form $\neg D$. Thus $\alpha_1^{\#} = \neg A$ and therefore $Q_i^{\#}$ results from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_{\to}$. Now suppose that $A = \neg D$. Hence $\alpha_1^{\#} = D$. However, in this case the following transformation via $\mathbf{E}^{\mathbf{RPQ}}$ leads to $Q_i^{\#}$:

$\mathbf{T}(\mathbf{L}_{\neg\to} / \boldsymbol{R}_{\to}, \boldsymbol{R}_{\neg\neg})$:

$? (\Phi; \vdash S^{\#}\text{'}\neg D \to C\text{'}T^{\#}\text{'}B; \Psi)$ $\quad \boldsymbol{R}_{\to}$

$? (\Phi; \vdash S^{\#}\text{'}\neg\neg D\text{'}C\text{'}T^{\#}\text{'}B; \Psi)$ $\quad \boldsymbol{R}_{\neg\neg}$

$? (\Phi; \vdash S^{\#}\text{'}D\text{'}C\ \text{'}T^{\#}\text{'}B; \Psi)$

Assume that $\alpha = \neg(A \vee B)$. Thus $\alpha^{\#} = A \vee B$, $\alpha_1^{\#} = A$, and $\alpha_2^{\#} = B$. Therefore $Q_i^{\#}$ results from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_{\vee}$.

<u>*Case 1.4*</u>: Q_i arises from $Q_{i\text{-}1}$ by $\mathbf{R}_{\beta}$. Hence $Q_{i\text{-}1}^{\#} = ? (\Phi; \vdash S^{\#}\text{'}\beta; \Psi)$ and $Q_i^{\#} = ? (\Phi; \vdash S^{\#}\text{'}(\beta_1{*})^{\#}\text{'}\beta_2; \Psi)$.

Assume that $\beta = \neg(A \wedge C)$. Thus $\beta_1{*} = A$. Suppose that A is not of the form $\neg D$. Hence $(\beta_1{*})^{\#} = \neg A$ and therefore we get $Q_i^{\#}$ from $Q_{i\text{-}1}^{\#}$ by $\mathbf{R}_{\neg\wedge}$. Suppose that $A = \neg D$. In this case $\beta_1{*} = \neg D$ and hence $(\beta_1{*})^{\#} = D$. However, the following transformation via $\mathbf{E}^{\mathbf{RPQ}}$ has the desired property:

$\mathbf{T}(\mathbf{R}_{\neg\wedge} / \boldsymbol{R}_{\wedge}, \boldsymbol{R}_{\neg\neg})$:

$? (\Phi; \vdash S^{\#}\text{'}\neg(\neg D \wedge C); \Psi)$ $\quad \boldsymbol{R}_{\beta}$

$? (\Phi; \vdash S\#\text{'}\neg\neg D\text{'}\neg C; \Psi)$ $\quad \boldsymbol{R}_{\neg\neg}$

$? (\Phi; \vdash S\#\text{'}D\text{'}\neg C; \Psi)$

Assume that $\beta = A \vee C$. Thus $\beta^{\#} = \neg(A \vee C)$. Suppose that $A = \neg D$. It follows that $\beta_1{*} = \neg\neg D$ and $(\beta_1{*})^{\#} = \neg D = A$. Therefore we get $Q_i^{\#}$ from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_{\vee}$. Now suppose that A is not of the form $\neg D$. In this case we have $\beta_1{*} = \neg A$ and hence $(\beta_1{*})^{\#} = A$. Again, we get $Q_i^{\#}$ from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_{\vee}$.

Assume that $\beta = A \to C$. Hence $\beta_1{*} = A$. Suppose that A is not of the form $\neg D$. Thus $(\beta_1{*})^{\#} = \neg A$ and therefore $Q_i^{\#}$ arises from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_{\to}$. Now suppose that $A = \neg D$. Since $\beta_1{*} = A$, it follows that $(\beta_1{*})^{\#} = D$. However, the following transformation via $\mathbf{E}^{\mathbf{RPQ}}$ leads to $Q_i^{\#}$:

T($\mathbf{R}_\rightarrow$ / $\boldsymbol{R}_{\neg\rightarrow}$, $\boldsymbol{R}_{\neg\neg}$):

$$? (\Phi;\ \vdash S^{\#}\ '(\neg D \rightarrow C);\ \Psi) \qquad \boldsymbol{R}_\beta$$

$$? (\Phi;\ \vdash S^{\#}\ '\neg\neg D\ 'C;\ \Psi) \qquad \boldsymbol{R}_{\neg\neg}$$

$$? (\Phi;\ \vdash S^{\#}\ 'D\ 'C;\ \Psi)$$

Case 1.5: Q_i arises from $Q_{i\text{-}1}$ by $\mathbf{R}_\alpha$. Since an application of $\mathbf{R}_\alpha$ has an analogous effect as an application of $\boldsymbol{R}_\alpha$, $Q_i^{\#}$ results from $Q_{i\text{-}1}^{\#}$ by $\boldsymbol{R}_\alpha$.

Case 1.6: Q_i arises from $Q_{i\text{-}1}$ by $\mathbf{R}_{\neg\neg}$. Again, the case is obvious (for similar reasons as above).

II. Quantificational cases

Case 2.1: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{L}_\forall$. Thus $Q_{i\text{-}1}^{\#}$ = ? (Φ; $\vdash S^{\#}\ '(\forall x_i\ A)^{\#}\ 'T^{\#}\ 'B$; Ψ) and $Q_i^{\#}$ = ? (Φ; $\vdash S^{\#}\ '(\forall x_i\ A)^{\#}\ '\ (A(x_i/\tau))^{\#}\ 'T^{\#}\ 'B$; Ψ).

Observe that $(\forall x_i\ A)^{\#} = \neg\forall x_i\ A$. If A is not of the form $\neg D$, then $(A(x_i/\tau))^{\#} = \neg A(x_i/\tau)$. The following transformation via $\mathbf{E}^{\mathbf{RPQ}}$ leads to $Q_i^{\#}$:

T₁($\mathbf{L}_\forall$ / $\boldsymbol{R}_{\underline{\mathbf{oqe}}}$, $\boldsymbol{R}_\exists$, $\boldsymbol{R}_{\underline{\mathbf{oqe}}}$):

$$? (\Phi;\ \vdash S^{\#}\ '\neg\forall x_i\ A\ 'T^{\#}\ 'B;\ \Psi) \qquad \boldsymbol{R}_{\underline{oqe}}$$

$$? (\Phi;\ \vdash S^{\#}\ '\exists x_i\ \neg A\ 'T^{\#}\ 'B;\ \Psi) \qquad \boldsymbol{R}_\exists$$

$$? (\Phi;\ \vdash S^{\#}\ '\exists x_i\ \neg A\ '\neg A(x_i/\tau)\ 'T^{\#}\ 'B;\ \Psi) \qquad \boldsymbol{R}_{\underline{oqe}}$$

$$? (\Phi;\ \vdash S^{\#}\ '\neg\forall x_i A\ '\neg A(x_i/\tau)\ 'T^{\#}\ 'B;\ \Psi)$$

If $A = \neg D$, then $(A(x_i/\tau))^{\#} =\ D(x_i/\tau)$. In this case we have the following transformation via $\mathbf{E}^{\mathbf{RPQ}}$ with the desired property:

T₂($\mathbf{L}_\forall$ / $\boldsymbol{R}_{\underline{\mathbf{oqe}}}$, $\boldsymbol{R}_\exists$):

$$? (\Phi;\ \vdash S^{\#}\ '\neg\forall x_i\ \neg D\ 'T^{\#}\ 'B;\ \Psi) \qquad \boldsymbol{R}_{\underline{oqe}}$$

$$? (\Phi;\ \vdash S^{\#}\ '\exists x_i\ D\ 'T^{\#}\ 'B;\ \Psi) \qquad \boldsymbol{R}_\exists$$

$$? (\Phi;\ \vdash S^{\#}\ '\exists x_i\ D\ 'D(x_i/\tau)\ 'T^{\#}\ 'B;\ \Psi) \qquad \boldsymbol{R}_{\underline{oqe}}$$

$$? (\Phi;\ \vdash S^{\#}\ '\neg\forall x_i\ \neg D\ 'D(x_i/\tau)\ 'T^{\#}\ 'B;\ \Psi)$$

Case 2.2: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{L}_\exists$. Hence $Q_{i\text{-}1}^{\#}$ = ? (Φ; $\vdash S^{\#}\ '(\exists x_i\ A)^{\#}\ 'T^{\#}\ 'B$; Ψ) and $Q_i^{\#}$ = ? (Φ; $\vdash S^{\#}\ '(A(x_i/\tau))^{\#}\ 'T^{\#}\ 'B$; Ψ).

Clearly, $(\exists x_i\ A)^{\#} = \neg\exists x_i\ A$. If A is not of the form $\neg D$, then $(A(x_i/\tau))^{\#} = \neg A(x_i/\tau)$. We get $Q_i^{\#}$ from $Q_{i\text{-}1}^{\#}$ as follows:

T₁($\mathbf{L}_\exists$ / $\mathbf{R}_{\underline{\mathbf{oqe}}}$, $\boldsymbol{R}_\forall$):

$$? (\Phi;\ \vdash S^{\#}\ '\neg\exists x_i\ A\ 'T^{\#}\ 'B;\ \Psi) \qquad \boldsymbol{R}_{\underline{oqe}}$$

$$? (\Phi;\ \vdash S^{\#}\ '\forall x_i\ \neg A\ 'T^{\#}\ 'B;\ \Psi) \qquad \boldsymbol{R}_\forall$$

$$? (\Phi;\ \vdash S^{\#}\ '\neg A(x_i/\tau)\ 'T^{\#}\ 'B;\ \Psi)$$

If $A = \neg D$, then $(A(x_i/\tau))^{\#} =\ D(x_i/\tau)$. In this case we have:

T₂($\mathbf{L}_\exists$ / $\mathbf{R}_{\underline{\mathbf{oqe}}}$, $\boldsymbol{R}_\forall$):

$$? (\Phi;\ \vdash S^{\#}\ '\neg\exists x_i\ \neg D\ 'T^{\#}\ 'B;\ \Psi) \qquad \boldsymbol{R}_{\underline{oqe}}$$

$? (\Phi;\ \vdash S^{\#}\ '\forall x_i\ D\ 'T^{\#}\ 'B;\ \Psi)$ **$R_\forall$**

$? (\Phi;\ \vdash S^{\#}\ 'D(x_i/\tau)\ 'T^{\#}\ 'B;\ \Psi)$

<u>*Case 2.3*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{L}_\kappa$ applied with respect to a sentence of the form $\forall x_i\ A$, where x_i is not free in A. Thus $Q_{i\text{-}1}^{\#} = ?\ (\Phi;\ \vdash S^{\#}\ `(\forall x_i\ A)^{\#}\ 'T^{\#}\ 'B;\ \Psi)$ and $Q_i^{\#} = ?\ (\Phi;\ \vdash S^{\#}\ '\ (A)^{\#}\ 'T^{\#}\ 'B;\ \Psi)$.

As above, we have $(\forall x_i\ A)^{\#} = \neg\forall x_i\ A$. There are two possibilities: (a) A is not of the form $\neg D$ and thus $(A)^{\#} = \neg A$, and (b) $A = \neg D$ and hence $(A)^{\#} = D$. Suppose that (a) holds. In this case we have:

$\mathbf{T_1}(\mathbf{L}_{\kappa\,(\forall*)}\,/\,\boldsymbol{R}_{\underline{\mathbf{oqe}}}\,,\,\boldsymbol{R}_{\exists*})$:

$? (\Phi;\ \vdash S^{\#}\ '\neg\forall x_i\ A\ 'T^{\#}\ 'B;\ \Psi)$ **$R_{\underline{oqe}}$**

$? (\Phi;\ \vdash S^{\#}\ '\exists x_i\ \neg A\ 'T^{\#}\ 'B;\ \Psi)$ **$R_{\exists*}$**

$? (\Phi;\ \vdash S^{\#}\ '\neg A\ 'T^{\#}\ 'B;\ \Psi)$

Now suppose that (b) takes place. The following transformation via $\mathbf{E}^{\mathbf{RPQ}}$ has the desired property:

$\mathbf{T_2}(\mathbf{L}_{\kappa\,(\forall*)}\,/\,\boldsymbol{R}_{\underline{\mathbf{oqe}}}\,,\,\boldsymbol{R}_{\exists*})$:

$? (\Phi;\ \vdash S^{\#}\ '\neg\forall x_i\ \neg D\ 'T^{\#}\ 'B;\ \Psi)$ **$R_{\underline{oqe}}$**

$? (\Phi;\ \vdash S^{\#}\ '\exists x_i\ D\ 'T^{\#}\ 'B;\ \Psi)$ **$R_{\exists*}$**

$? (\Phi;\ \vdash S^{\#}\ 'D\ 'T^{\#}\ 'B;\ \Psi)$

<u>*Case 2.4*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{L}_\kappa$ applied with respect to a sentence of the form $\exists x_i\ A$, where x_i is not free in A. We reason analogously as above and come to the conclusion that $Q_i^{\#}$ can be reached from $Q_{i\text{-}1}^{\#}$.

<u>*Case 2.5*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{L}_{\neg\exists}$. It is clear that $Q_i^{\#}$ arises from $Q_{i\text{-}1}^{\#}$ by **$R_{\underline{oqe}}$**.

<u>*Case 2.6*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{L}_{\neg\forall}$. Again, it is obvious that $Q_i^{\#}$ arises from $Q_{i\text{-}1}^{\#}$ by **$R_{\underline{oqe}}$**.

<u>*Case 2.7*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{R}_\forall$. Now $Q_i^{\#}$ arises from $Q_{i\text{-}1}^{\#}$ by **$R_\forall$**.

<u>*Case 2.8*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{R}_\exists$. Thus $Q_{i\text{-}1}^{\#} = ?\ (\Phi;\ \vdash S^{\#}\ `\exists x_i A;\ \Psi)$ and $Q_i^{\#} = ?\ (\Phi;\ \vdash S^{\#}\ `\neg\forall x_i \neg A\ `A(x_i/\tau);\ \Psi)$. We arrive at $Q_i^{\#}$ in the following transformation via $\mathbf{E}^{\mathbf{RPQ}}$:

$\mathbf{T}(\mathbf{L}_\exists/\boldsymbol{R}_\exists\,,\,\boldsymbol{R}_{\underline{\mathbf{oqe}}})$:

$? (\Phi;\ \vdash S^{\#}\ '\exists x_i A;\ \Psi)$ **$R_\exists$**

$? (\Phi;\ \vdash S^{\#}\ '\exists x_i A\ 'A(x_i/\tau);\ \Psi)$ **$R_{\underline{oqe}}$**

$? (\Phi;\ \vdash S^{\#}\ '\neg\forall x_i \neg A\ 'A(x_i/\tau);\ \Psi)$

<u>*Case 2.9*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{R}_\kappa$ applied with respect to a sentence of the form $\forall x_i\ A$, where x_i is not free in A. Now $Q_i^{\#}$ arises from $Q_{i\text{-}1}^{\#}$ by **$R_{\forall*}$**.

<u>*Case 2.10*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{R}_\kappa$ applied with respect to a sentence of the form $\exists x_i\ A$, where x_i is not free in A. It is clear that $Q_i^{\#}$ comes from $Q_{i\text{-}1}^{\#}$ by **$R_{\exists*}$**.

<u>*Case 2.11*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{R}_{\neg\exists}$. It is obvious that $Q_i^{\#}$ arises from $Q_{i\text{-}1}^{\#}$ by **$R_{\underline{oqe}}$**.

<u>*Case 2.12*</u>: Q_i results from $Q_{i\text{-}1}$ by rule $\mathbf{R}_{\neg\forall}$. Now we get $Q_i^{\#}$ from $Q_{i\text{-}1}^{\#}$ by **$R_{\underline{oqe}}$**.

■

Lemma 3: *Let* $\mathbf{s} = \langle Q_1, ..., Q_n \rangle$ *be a Socratic proof of* $\vdash A$ *in* $\mathbf{E}^{\mathbf{PQ}}$ *and let* $Q_n = ?\ (S_1 \vdash A_1, \ldots, S_n \vdash A_n)$. *Let* $Q_n^{\#} = ?\ (\vdash S_1^{\#}\,'A_1, \ldots, \vdash S_n^{\#}\,'A_n)$. *Each constituent of* $Q_n^{\#}$ *is successful, i.e. is of the form* $\vdash T\,'B\,'U\,'\neg B\,'W$, *or is of the form* $\vdash T\,'\neg B\,'U\,'B\,'W$.

P r o o f: Since Q_n is the last question of a Socratic proof in $\mathbf{E}^{\mathbf{PQ}}$, then for each constituent of Q_n at least one of the following holds: (a) there is a sentence which occurs both left and right of the turnstile, (b) there is a sentence such that this sentence and its negation occurs left of the turnstile. Thus, by definitions 4, 5, and 6, each constituent of $Q_n^{\#}$ is successful. ■

Lemma 4. *If* $\vdash A$ *is provable in* $\mathbf{E}^{\mathbf{PQ}}$, *then* $\vdash A$ *is provable in* $\mathbf{E}^{\mathbf{RPQ}}$.

P r o o f: Let $\mathbf{s} = \langle Q_1, ..., Q_n \rangle$ be a Socratic proof of $\vdash A$ in $\mathbf{E}^{\mathbf{PQ}}$. We consider the following sequence $\mathbf{s}^{\#}$ of questions of $\boldsymbol{L}^{**}$

(2) $\langle Q_1^{\#}, ..., Q_n^{\#} \rangle$

According to Definition 4, $Q_1^{\#} = ?\ (\vdash A)$. By Lemma 3, each constituent of $Q_n^{\#}$ is successful. By Lemma 2, for each index i such that $1 < i \leq n$ we have:

(a) $Q_i^{\#} = Q_{i-1}^{\#}$, or

(b) $Q_i^{\#}$ results from $Q_{i-1}^{\#}$ by a rule of $\mathbf{E}^{\mathbf{RPQ}}$, or

(c) $Q_i^{\#}$ does not result from $Q_{i-1}^{\#}$ by a rule of $\mathbf{E}^{\mathbf{RPQ}}$, but there exists a finite Socratic transformation of $Q_{i-1}^{\#}$ via $\mathbf{E}^{\mathbf{RPQ}}$ that leads to $Q_i^{\#}$.

Now observe that the above conditions are mutually exclusive. So for a given index i ($1 < i \leq n$) exactly one of them is fulfilled.

If condition (b) holds for each index i such that $1 < i \leq n$, then $\mathbf{s}^{\#}$ is a Socratic proof of $\vdash A$ in $\mathbf{E}^{\mathbf{RPQ}}$. Otherwise we take $\mathbf{s}^{\#}$ and we act as follows:

(*) we delete consecutive occurrences of the same question, *i.e.* if $Q_i^{\#} = Q_{i-1}^{\#}$, then we delete $Q_i^{\#}$ and leave $Q_{i-1}^{\#}$ only, and/or

(**) we embed the appropriate Socratic transformation of $Q_{i-1}^{\#}$ that leads to $Q_i^{\#}$, according to the schemata presented in the proof of Lemma 2, *i.e.* if $Q_i^{\#}$ does not result from $Q_{i-1}^{\#}$ by a rule of $\mathbf{E}^{\mathbf{RPQ}}$ and $\langle Q_{i-1}^{\#}, Q_1', ..., Q_k', Q_i^{\#} \rangle$ is the Socratic transformation via $\mathbf{E}^{\mathbf{RPQ}}$ that leads to $Q_i^{\#}$, we replace the subsequence $\langle Q_{i-1}^{\#}, Q_i^{\#} \rangle$ with $\langle Q_{i-1}^{\#}, Q_1', ..., Q_k', Q_i^{\#} \rangle$ (observe that $0 < k \leq 2$).

It is clear that a sequence obtained from $\mathbf{s}^{\#}$ in the above manner is a Socratic transformation of $?\ (\vdash A)$ via $\mathbf{E}^{\mathbf{RPQ}}$. Since $Q_n^{\#}$ is still the last question of this sentence[5] and $Q_n^{\#}$ involves only successful constituents, the outcome is a Socratic proof of $\vdash A$ in $\mathbf{E}^{\mathbf{RPQ}}$. ■

Theorem 2 (completeness of $\mathbf{E}^{\mathbf{RPQ}}$): *Let A be a parameter-free sentence of* $\mathbf{L}$. *If A is valid, then* $\vdash A$ *is provable in* $\mathbf{E}^{\mathbf{RPQ}}$.

P r o o f: If A is valid, then the sequent $\vdash A$ is valid. Due to the completeness of $\mathbf{E}^{\mathbf{PQ}}$, $\vdash A$ is provable in $\mathbf{E}^{\mathbf{PQ}}$. Therefore, by Lemma 4, $\vdash A$ is provable in $\mathbf{E}^{\mathbf{RPQ}}$ as well. ■

5. Final Remarks

The calculus $\mathbf{E}^{\mathbf{RPQ}}$ presented here originated from work on erotetic calculi for FOL in which questions are based on single-conclusioned sequents (and thus an "operative" interpre-

[5] Even if action (*) was taken with respect to the last question of $\mathbf{s}^{\#}$, because this can happen only if this question is identical with the previous one.

tation of the turnstile is possible), and which is grounded in Inferential Erotetic Logic. The calculus $\mathbf{E}^{\mathbf{PQ}}$ mentioned above was the result of the enterpise. $\mathbf{E}^{\mathbf{PQ}}$ can be easily transformed into a (non-standard) calculus of hypersequents; moreover, it determines a certain Gentzen-style calculus (for details, see Wiśniewski and Shangin 2006). As long as $\mathbf{E}^{\mathbf{RPQ}}$ is concerned, a "translation" of $\mathbf{E}^{\mathbf{RPQ}}$ into a variant of Rasiowa-Sikorski calculus for FOL (see Rasiowa and Sikorski 1960) is almost immediate: by and large, it suffices to remove turnstiles and question marks from the rules. The rule resulting from $\boldsymbol{R}_{\underline{\mathbf{oqe}}}$, however, licenses some transitions which are not licensed by the original Rasiowa-Sikorski system (where transitions between $\nabla x_i\ A$ and $\neg\Delta x_i\ \neg A$ are not allowed, as well as transitions from $\Delta x_i\ \neg A$ to $\neg\nabla x_i\ A$). But the most interesting feature of Rasiowa-Sikorski style systems, that is, semantical invertibility of rules, is still retained. It seems that an "erotetic" calculus with a weaker version of $\boldsymbol{R}_{\underline{\mathbf{oqe}}}$ (that is, licensing only transitions from $\neg\nabla x_i\ A$ to $\Delta x_i\ \neg A$) is complete as well. We give rule $\boldsymbol{R}_{\underline{\mathbf{oqe}}}$ the current form because such a move facilitates translations of $\mathbf{E}^{\mathbf{PQ}}$-proofs into proofs dealing with right-sided sequents only and sheds some light on the problem of duality.

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