Being Permitted, Inconsistencies, and Question Raising

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Abstract. A semantic relation of being permitted by a set of possible worlds is defined and analysed. We call it "permittance". The domain of permittance comprises declarative sentences/formulas. A paraconsistent consequence relation which is both permittance-preserving and truth-preserving is characterized. An application of the introduced concepts in the analysis of question raising is presented.

Mathematics Subject Classification (2000). Primary 03B60; Secondary 03B20, 03B53.

Keywords. Being permitted, consequence, paraconsistency, Logic of Questions.

1. Introduction

We are often confronted with a number of alternative accounts of how things are, yet without knowing which of the accounts, if any, is the right one. These accounts disagree on some issues and agree on others. Despite discrepancies, however, some facts still remain known, some states of affairs are considered impossible, and some statements are *permitted* while other are not.

In this paper we define the relation "a declarative sentence is permitted by a set of possible worlds" and we analyse its basic properties. The possible worlds in question are supposed to represent alternative accounts of how things are. We dub the relation "permittance". The definition proposed is an explication of the corresponding intuitive notion of permitting, taken in one of its meanings. Our

To appear in: J.-Y. Beziau, M. Chakraborty and S. Dutta (eds.), *New Directions in Paraconsistent Logic*, Springer Proceedings in Mathematics and Statistics 152, 2015. DOI 10.1007/978-81-322-2719-9_22.

Work on this paper was supported by funds of the National Science Centre, Poland (DEC-2012/04/A/HS1/00715).) I am indebted to Dorota Leszczyńska-Jasion, Mariusz Urbański and Tomasz Skura for their comments. Some remarks of Ivano Ciardelli were also of help.

intuitions are presented in Section 1.2. For clarity, we start with a short description of the basic logical tools used throughout the paper.

1.1. Logical preliminaries

We remain at the propositional level only. We consider a non-modal propositional language, L. The vocabulary of L includes a non-empty set $\mathcal{P} = \{p, q, r, \ldots\}$ of propositional variables, the propositional constant $\bot$ (*falsum*), and the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$. Well-formed formulas (wffs for short) of L are defined in the usual manner. We shall use the letters A, B, C, ..., with subscripts if needed, as metalanguage variables for wffs of L. The letters X, Y, ... are metalanguage variables for sets of wffs of L.

The connectives, as well as $\bot$, are understood, at the truth-functional level, as in Classical Propositional Logic. By an *L-model* we mean an ordered pair $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$, where $\mathcal{W} \neq \emptyset$ and $\mathcal{V} : \mathcal{P} \times \mathcal{W} \longmapsto \{\mathbf{1}, \mathbf{0}\}$ is a valuation of propositional variables in $\mathcal{P}$ w.r.t. elements of $\mathcal{W}$. As usual, the elements of the domain, $\mathcal{W}$, are called possible worlds. The concept of truth of a wff A in a world $w \in \mathcal{W}$ of $\mathbf{M}$, in symbols $\mathbf{M}, w \models A$, is defined in the standard manner.The inscription $\mathbf{M} \models A$ means "A is true in $\mathbf{M}$", that is, A is true in each world of the domain of $\mathbf{M}$.

Elements of domains of L-models, the possible worlds, will be intuitively thought of here as *alternative accounts of how things are.* This has no impact on the formalism, however. As long as we remain at the propositional level, the only condition imposed on $\mathcal{W}$ is non-emptiness. It follows that the domain of an L-model need not contain all the relevant alternatives.

By a *state* we will mean a non-empty set of possible worlds. In view of the intuitive interpretation of possible worlds adopted above, a non-singleton state comprises a number of alternative accounts of how things are.

Let $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ be an L-model.

Definition 1 (Truth set of a wff in an L-model). $|A|_{\mathbf{M}} = \{w \in \mathcal{W} : \mathbf{M}, w \models A\}$.

Of course, $|\bot|_{\mathbf{M}} = \emptyset$.

Definition 2 (M-state). *An* $\mathbf{M}$*-state is a non-empty subset of* $\mathcal{W}$.

Note that $\mathcal{W}$ is (also) an $\mathbf{M}$-state, and that, for each $w \in \mathcal{W}$, the singleton set $\{w\}$ is an $\mathbf{M}$-state.

1.2. Intuitions

Our basic intuition concerning the analysed concept of being permitted is:

(I) *A declarative sentence/wff* γ *is permitted by a state* σ *iff it is not the case that* σ *rules out* γ.

However, what "rules out" means depends on the form of γ.

γ can be *positive* that is not of the form $\neg\zeta$ (where $\neg$ stands for sentential negation and ζ is a declarative sentence/wff). It is natural to postulate:

(II) *Let* γ *be positive. State* σ *rules out* γ *iff* γ *is false in each world of* σ.

For example, "Andrew is a bachelor" is ruled out by a state which comprises (only) possible worlds in which Andrew is married.

γ can be *negative* that is of the form $\neg\xi$, where ξ is positive.[1] We seem justified in saying:

(III) *Let γ be negative and $\gamma = \neg\xi$. State σ rules out γ iff ξ is true in some world of σ.*

For instance, a state that contains a possible world in which Andrew is a bachelor rules out the sentence "It is not the case that Andrew is a bachelor."

Assuming bivalence, by (I) and (II) we get:

(II*) *A positive, γ, is permitted by a state σ iff γ is true in some world of σ.*

By (I) and (III), in turn, we get:

(III*) *A negative, γ, is permitted by a state σ iff γ is true in each world of σ.*

An analogy can be of help. A civil servant is permitted to issue a positive decision if there is a rule that entitles him/her to do so, and is permitted to decide to the negative if the disputed activity is forbidden by each rule that is applicable to the case. Similarly, a negative is permitted by a state if there is no world of the state that makes the negated sentence true, while for a positive being permitted by a state amounts to the existence of a world of the state which makes it true. Our usage of "being permitted" is thus akin to that of its deontic cousin. Yet, we do not aim at analysing "being permitted" deontically construed. Permittance in our sense is a relation between a declarative sentence/wff on the one hand, and a state on the other. What is (or is not) permitted is a declarative sentence/wff, and what permits it (or does not permit) is a set of possible worlds, where possible worlds are intuitively thought of as alternative accounts of how things are.[2]

The paper is organized as follows. In Section 2 we define the concept of permittance, characterize its basic properties, and show how knowledge and epistemic possibility can be modelled in our framework. Section 3 is devoted to permittance of inconsistencies. In Section 4 we analyse a paraconsistent consequence relation of transmission of permittance. Section 5 addresses the issue of question raising, in particular the problem of question raising by inconsistencies.

2. Permittance

2.1. Definition and basic properties

Let $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ be an arbitrary but fixed L-model. "$\sigma \looparrowright A$" reads: "wff A is permitted by an $\mathbf{M}$-state σ". "$\looparrowright$" is thus the sign of the permittance relation.

[1] Observe that $\neg\neg\xi$ is neither negative nor positive. We will come back to this issue later on.

[2] Looking from a formal point of view, permittance belongs to the same category as *support* analysed in Inquisitive Semantics (see, e.g., [1], [7], [2]). However, the underlying intuitions are different. Moreover, Inquisitive Semantics conceives states/sets of possible worlds as information states.

Given the considerations presented above, the following definition comes with no surprise.

Definition 3 (Permittance).

1. $\sigma \looparrowright \mathbf{p}$ *iff* $|\mathbf{p}|_{\mathbf{M}} \cap \sigma \neq \emptyset$, *for any propositional variable* $\mathbf{p}$;
2. $\sigma \looparrowright \neg A$ *iff* $\sigma \not\looparrowright A$;
3. $\sigma \looparrowright (A \vee B)$ *iff* $|(A \vee B)|_{\mathbf{M}} \cap \sigma \neq \emptyset$;
4. $\sigma \looparrowright (A \wedge B)$ *iff* $|(A \wedge B)|_{\mathbf{M}} \cap \sigma \neq \emptyset$;
5. $\sigma \looparrowright (A \to B)$ *iff* $|(A \to B)|_{\mathbf{M}} \cap \sigma \neq \emptyset$;
6. $\sigma \looparrowright \bot$ *iff* $|\bot|_{\mathbf{M}} \cap \sigma \neq \emptyset$.

Observe that permittance *is not* defined inductively. This is intended.

For positive wffs, being permitted by a state amounts to being true in some world(s) of the state. To be more precise, as an immediate consequence of Definition 3 we get:

Corollary 1. *Let* σ *be an* $\mathbf{M}$*-state and let* A *be a positive wff. Then* $\sigma \looparrowright A$ *iff* $\mathbf{M}, w \models A$ *for some* $w \in \sigma$.

However, the case of negative wffs is different. By Corollary 1 and clause (2) of Definition 3 we have:

Corollary 2. *Let* σ *be an* $\mathbf{M}$*-state. Let* D *be a wff of any of the forms:* $\mathtt{p}$, $\bot$, $(B \vee C)$, $(B \wedge C)$, $(B \to C)$. *Then* $\sigma \looparrowright \neg D$ *iff* $\mathbf{M}, w \not\models D$ *for each* $w \in \sigma$.

Hence:

Corollary 3. *Let* σ *be an* $\mathbf{M}$*-state and let* A *be a negative wff. Then* $\sigma \looparrowright A$ *iff* $\mathbf{M}, w \models A$ *for each* $w \in \sigma$.

Corollary 3 shows that negatives behave in the context of permittance as it has been required in section 1.2.

But what about wffs which are neither positive nor negative? As for L, there is only one kind of such wffs, namely wffs falling under the general schema:

$$\neg \ldots \neg D \tag{1}$$

where D is positive and the number of negations preceding D is greater than 1. If the number is even, we say that (1) is a $\neg_e$-wff; otherwise (1) is a $\neg_o$-wff. By D_A we designate the positive wff which occurs in a $\neg_e$-wff A or in a $\neg_o$-wff A after the string of negations.[3]

One can prove:

Corollary 4. $\sigma \looparrowright \neg\neg A$ *iff* $\sigma \looparrowright A$.

Proof. By the clause (2) of Definition 3 we have:

$$\sigma \looparrowright \neg\neg A \text{ iff } \sigma \not\looparrowright \neg A$$
$$\sigma \not\looparrowright \neg A \text{ iff } \sigma \looparrowright A$$

and hence $\sigma \looparrowright \neg\neg A$ iff $\sigma \looparrowright A$. □

[3]When A is neither positive nor negative, D_A is in the scope of the rightmost negation of the string.

Thus, taking into account corollaries 1, 2, and 4 we get:

Corollary 5.

1. *Let A be a $\neg_e$-wff. Then $\sigma \looparrowright A$ iff $\mathbf{M}, w \models D_A$ for some $w \in \sigma$ iff $\mathbf{M}, w \models A$ for some $w \in \sigma$.*
2. *Let A be a $\neg_o$-wff. Then $\sigma \looparrowright A$ iff $\mathbf{M}, w \not\models D_A$ for each $w \in \sigma$ iff $\mathbf{M}, w \models A$ for each $w \in \sigma$.*

For brevity, let us introduce:

Definition 4 (p-wffs and n-wffs).

1. *A p-wff is a positive wff or a $\neg_e$-wff.*
2. *A n-wff is a negative wff or a $\neg_o$-wff.*

As we have shown, the categories of p-wffs and n-wffs are semantically homogeneous: a p-wff is permitted by a state iff it is true in at least one world of the state, while a n-wff is permitted by a state iff it is true in each world of the state. Permittance could had been concisely defined in terms of p-wffs and n-wffs. However, doing this would require an ad hoc acceptance of the claim of Corollary 4.

2.1.1. Remarks.

Remark 1. For a singleton state permittance amounts to truth in the only world of the state. As an immediate consequence of the above corollaries we get:

Corollary 6. *Let $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ be an L-model and $\{w\}$ be a (singleton) $\mathbf{M}$-state. Then $\{w\} \looparrowright A$ iff $\mathbf{M}, w \models A$.*

Remark 2. Permittance becomes intensional when non-singleton states enter the picture. It happens that wffs which have equal truth sets (i.e. are classically equivalent) are not simultaneously permitted by a state. For example, we have:

$$|\neg(p \to q)|_{\mathbf{M}} = |p \wedge \neg q|_{\mathbf{M}}$$

Now take an L-model and its state $\{w_1, w_2\}$ such that:

- $\mathcal{V}(p, w_1) = \mathbf{1}$ and $\mathcal{V}(q, w_1) = \mathbf{0}$,
- $\mathcal{V}(p, w_2) = \mathbf{0}$ and $\mathcal{V}(q, w_2) = \mathbf{0}$.

We get:

$$\{w_1, w_2\} \not\looparrowright \neg(p \to q)$$
$$\{w_1, w_2\} \looparrowright p \wedge \neg q$$

Remark 3. Note that wffs of the forms:

$$\neg A \tag{2}$$
$$A \to \bot \tag{3}$$

do no differ as to their truth conditions in a world, but can differ with respect to permittance by states. When A is a p-wff, (2) is permitted only by a state in which A is false in each world of the state, whereas (3) can be permitted by a state in which A is false only in some, but not all worlds. This does not mean, however, that the negation connective $\neg$ has a non-classical meaning in L. Its meaning is determined by the standard truth condition. But $\neg$ behaves in a somewhat non-standard way in the context of permittance.

Remark 4. Observe that for any wff A, any L-model $\mathbf{M}$, and any $\mathbf{M}$-state σ we have:

$$\sigma \looparrowright (\neg A \to \bot) \text{ iff } \sigma \looparrowright A \wedge A \tag{4}$$

Hence we are able to express in terms of permittance, and without using $\bot$, that a n-wff, B, is true in at least one, but not necessarily all worlds of a state σ; this holds just in case the wff $B \wedge B$ is permitted by σ.

Note also that in general permittance is neither downward closed (if A is a p-wff, permittance of A by σ need not yield permittance of A by a proper subset of σ) nor upward closed (a n-wff permitted by a state need not be permitted by an extension of the state). However, permittance is upward closed for p-wffs and downward closed in the case of n-wffs.

2.2. Modalization

Let us now augment the initial language L with the modalities $\Box$ (necessity) and $\Diamond$ (possibility). Wffs of the enriched language are defined in the standard manner. We label the new language as $\mathcal{L}$. We use $\phi, \psi, \ldots$ as metalanguage variables for wffs of $\mathcal{L}$, and $\Phi, \Psi, \ldots$ as metalanguage variables for sets of wffs of the language. Whenever $\Box$ or $\Diamond$ precedes a metalanguage expression referring to wffs of L, it is understood that the wff in the scope of a modality belongs to L (i.e. is a wff of $\mathcal{L}$ in which no modality occurs).

Definition 5 (S5-model). *An* **S5***-model is a structure:*

$$\langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$$

where $\mathcal{W} \neq \emptyset$, $\mathcal{V}$ is a valuation of $\mathcal{P}$ w.r.t. elements of $\mathcal{W}$, and $\mathcal{R} = \mathcal{W} \times \mathcal{W}$.

Thus by **S5**-models we will mean here only these relational models in which the accessibility relation $\mathcal{R}$ is universal. In the case of **S5**-models we have:

$$\mathcal{M}, w \models \Box\phi \textit{ iff } \mathcal{M}, w \models \phi \textit{ for each } w \in \mathcal{W}, \tag{5}$$

$$\mathcal{M}, w \models \Diamond\phi \textit{ iff } \mathcal{M}, w \models \phi \textit{ for some } w \in \mathcal{W}. \tag{6}$$

It is well-known that **S5** is sound and complete w.r.t. the class of models of the above kind.

Definition 6 (Accompanied S5-model). *Let $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ be an L-model, σ be an $\mathbf{M}$-state, and $\mathcal{R} = \mathcal{W} \times \mathcal{W}$. Let $\mathcal{M}_\sigma$ be an* **S5***-model such that:*

$$\mathcal{M}_\sigma = \langle \sigma, \mathcal{R}|\sigma, \mathcal{V}|\sigma \rangle$$

$\mathcal{M}_\sigma$ is called the **S5***-model accompanied with $\mathbf{M}$ w.r.t. state σ.*

It is obvious that for each L-model $\mathbf{M}$ and each state of the model there exists exactly one **S5**-model accompanied with $\mathbf{M}$ w.r.t. the state. For each wff A of L we have:

Corollary 7. *Let* $\mathbf{M}$ *be an* L*-model,* σ *be an* $\mathbf{M}$*-state, and* $w \in \sigma$*. Then* $\mathbf{M}, w \models A$ *iff* $\mathcal{M}_\sigma, w \models A$*.*

The following is true as well:

Lemma 1. *For each* $\mathbf{M}$*-state* σ*:*

1. *if* A *is a p-wff, then:* $\sigma \looparrowright A$ *iff* $\mathcal{M}_\sigma \models \Diamond A$*,*
2. *if* A *is a n-wff, then:* $\sigma \looparrowright A$ *iff* $\mathcal{M}_\sigma \models \Box A$*.*

Proof. As for (1), it suffices to recall that for a p-wff A we have $\sigma \looparrowright A$ iff A is true in at least one world of σ. On the other hand, the accessibility relation in $\mathcal{M}_\sigma$ is universal and thus $\mathcal{M}_\sigma \models \Diamond A$ iff $\mathcal{M}_\sigma, w \models A$ for at least one $w \in \sigma$.

Clause (2) is an immediate consequence of Corollary 3 and the fact that $\mathcal{R}|\sigma$ is universal in σ. □

Let us also prove:

Lemma 2. *Let* A *be a wff of* L*. For each* $\mathbf{M}$*-state* σ*:*

1. $\sigma \looparrowright \neg(A \to \bot)$ *iff* $\mathcal{M}_\sigma \models \Box A$*,*
2. $\sigma \looparrowright (A \to \bot)$ *iff* $\mathcal{M}_\sigma \models \Diamond\neg A$*,*
3. $\sigma \looparrowright (\neg A \to \bot)$ *iff* $\mathcal{M}_\sigma \models \Diamond A$*.*

Proof. As for (1), $\neg(A \to \bot)$ is a n-wff and hence, by Corollary 3, $\sigma \looparrowright \neg(A \to \bot)$ iff for each $w \in \sigma$: $\mathbf{M}, w \not\models (A \to \bot)$, that is, $\mathcal{M}_\sigma, w \models A$ for any $w \in \sigma$, which, due to the universality of $\mathcal{R}|\sigma$ gives $\mathcal{M}_\sigma \models \Box A$.

Concerning (2): $\sigma \looparrowright (A \to \bot)$ iff $|(A \to \bot)|_{\mathbf{M}} \cap \sigma \neq \emptyset$ iff for some $w \in \sigma$: $\mathbf{M}, w \models \neg A$ iff $\mathcal{M}_\sigma \models \Diamond\neg A$.

(3) is a direct consequence of (2). □

2.3. Epistemization

As it is well-known, **S5** can be interpreted as an epistemic logic, where the box, $\Box$, represents the knowledge operator, and the diamond, $\Diamond$, represents, generally speaking, epistemic possibility. This suggests a kind of purely epistemic readings of some *metalanguage* expressions of the form "$\sigma \looparrowright A$".

Consider:

$$\sigma \looparrowright \neg(A \to \bot) \tag{7}$$

Due to clause (1) of Lemma 2, this can be read:

$$A \textit{ is known in state } \sigma \tag{8}$$

where "A *is known in state* σ" means:

$$\mathcal{M}_\sigma \models \Box A \tag{9}$$

that is, $\Box A$ is true in an **S5**-model whose domain is σ (more precisely: $\Box A$ is true in the **S5**-model whose domain is the $\mathbf{M}$-state σ and which agrees with $\mathbf{M}$ on the values of propositional variables w.r.t. worlds in σ).

Observe that, by Lemma 2, "being known in σ" does not differentiate between n-wffs and p-wffs.

Example 1. Let us consider the case of implication. In our setting $(A \to B)$ is said to be known in a state σ iff:

$$\sigma \looparrowright \neg((A \to B) \to \bot) \tag{10}$$

It follows that:

$$\mathcal{M}_\sigma \models \Box(A \to B) \tag{11}$$

and:

$$\textit{for each } w \in \sigma : \mathbf{M}, w \models (A \to B) \tag{12}$$

Thus an implication constitutes an item of knowledge in a state if, and only if it is true in each world of the state. Or, to put it differently, an implication is known in a state just in case it is a strict implication w.r.t. the state.

Example 2. Now consider the case in which the negation of an implication, i.e. $\neg(A \to B)$, is known in state σ. This means:

$$\sigma \looparrowright \neg(\neg(A \to B) \to \bot) \tag{13}$$

which gives:

$$\mathcal{M}_\sigma \models \Box\neg(A \to B) \tag{14}$$

Hence:

$$\textit{for each } w \in \sigma : \mathbf{M}, w \not\models (A \to B) \tag{15}$$

So a negated implication is an item of knowledge in a state just in case the implication itself is false in each world of the state. It follows that a negated implication is known in a state if, and only if it is permitted by the state.

Let us consider expressions of the form:

$$\sigma \looparrowright (A \to \bot) \tag{16}$$

By Lemma 2 we have:

$$\sigma \looparrowright (A \to \bot) \textit{ iff } \mathcal{M}_\sigma \models \Diamond\neg A \tag{17}$$

Hence an expression of the form (16) can be read:

$$\neg A \textit{ is epistemically possible in } \sigma \tag{18}$$

again uniformly for all the wffs of L.

Now let us consider:

$$\sigma \looparrowright (\neg A \to \bot) \tag{19}$$

By Lemma 2 we get:

$$\sigma \looparrowright (\neg A \to \bot) \textit{ iff } \mathcal{M}_\sigma \models \Diamond A \tag{20}$$

Thus one can read (19) as:

$$A \textit{ is epistemically possible in } \sigma \tag{21}$$

For convenience, we introduce:

Definition 7 ("Epistemic" modalities).

1. $\boxplus A =_{df} \neg(A \to \bot)$
2. $\oplus A =_{df} (\neg A \to \bot)$

3. $\ominus A =_{df} (A \to \bot)$

Observe that $\boxplus$ *is not* the **S5** necessity/knowledge operator. Let A be a wff of L and let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ be an **S5**-model. Consider an arbitrary but fixed world $w \in \mathcal{W}$. Clearly, $\boxplus A$ is true in w iff A is true in w, while $\Box A$ is true in w just in case A is true in each world of $\mathcal{W}$. Thus $\boxplus A$ and $\Box A$ have different truth-conditions in worlds.[4] But $\boxplus$ behaves similarly as the **S5** knowledge operator. One can easily prove:

Corollary 8. *Let A, B be wffs of L.*

1. *The following:*

$$\boxplus(A \to B) \to (\boxplus A \to \boxplus B) \tag{22}$$

$$\boxplus A \to A \tag{23}$$

$$\neg \boxplus A \to \boxplus \neg \boxplus A \tag{24}$$

are true in each L-model.

2. *If A is true in each L-model, then $\boxplus A$ is true in each L-model.*

However, $\sigma \looparrowright \boxplus A$ only means $\mathcal{M}_\sigma \models \Box A$ (or equivalently: $\langle \sigma, \mathcal{V}|\sigma \rangle \models \neg(A \to \bot)$). Thus a wff known in a state σ of an L-model $\langle \mathcal{W}, \mathcal{V} \rangle$ must be true in each world of the state σ, but not necessarily in each world of the whole model. In other words, knowledge in a state is factive w.r.t. worlds of the state, but need not be factive with regard to all worlds of the model. Yet, when one considers a singleton state, it is impossible that a wff A is known in the state (in the sense of $\boxplus$) when A is false in the (only) world of the state.

2.3.1. A philosophical comment. The standard philosophical concept of knowledge conceives it as a *true* justified belief about the actual world. In the framework of an epistemic logic supplemented with a relational semantics "being known in a world w of a model" is explicated by "being true in each world w^* of the model such that w^* is accessible from w". When **S5** is used as an epistemic logic, this amounts to being true in each world of the model. Since we usually assume that the actual world is among the possible worlds considered (or is represented by a certain possible world of a model), the truth of $\Box A$ in a model yields the truth of A in the actual world, and $\Box A$ is true in the actual world only if A is true in the world.

Knowledge in a state behaves differently. If A is known in a state σ, it is true in each world of the state and thus also in the actual world *if* the actual world "is" in σ. This, however, need not be the case.

[4]However, since $\mathcal{R}$ is supposed to be universal, $\boxplus A$ in true in an L-model or in an **S5**-model iff $\Box A$ is true in the model(s).

3. Permittance and inconsistency

As above, we assume that $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ is an arbitrary but fixed L-model. The $\mathbf{M}$-*permittance class* of a wff A of L, in symbols: $\|A\|_{\mathbf{M}}$, comprises all the $\mathbf{M}$-states that permit A. The $\mathbf{M}$-permittance class of a set of wffs X, $\|X\|_{\mathbf{M}}$, in turn, is the intersection of $\mathbf{M}$-permittance classes of elements of X. More formally:

Definition 8 (Permittance class).

1. $\|A\|_{\mathbf{M}} = \{\sigma \subseteq \mathcal{W} : \sigma \neq \emptyset \text{ and } \sigma \looparrowright A\}$
2. $\|X\|_{\mathbf{M}} = \{\sigma \subseteq \mathcal{W} : \sigma \looparrowright B \text{ for each } B \in X\}$.

Definition 9. *X has a non-empty permittance class iff there exists an L-model $\mathbf{M}$ such that $\|X\|_{\mathbf{M}} \neq \emptyset$.*

When $\{A\}$ has a non-empty permittance class, we will be saying briefly: "A has a non-empty permittance class."

One can show that some inconsistent sets of wffs have non-empty permittance classes. For clarity, let us first introduce:

Definition 10 (Inconsistent sets). *A set of wffs X of L is:*

1. *inconsistent iff $\bigcap_{B \in X} |B|_{\mathbf{M}} = \emptyset$ for each L-model $\mathbf{M}$;*
2. *plainly inconsistent iff:*
   (a) *for some wff A, both $A \in X$ and $\ulcorner \neg A \urcorner \in X$, or*
   (b) *for some wff $A \in X$, $\{A\}$ is inconsistent.*

Clearly, permittance classes of plainly inconsistent sets are always empty. However, the situation is different in the case of some sets of wffs which are inconsistent, but not plainly inconsistent.

For example, $\{A, A \to \bot\}$ is inconsistent. But the following holds:

Corollary 9. *Let A be a p-wff of L such that $\ulcorner \Diamond A \to \Box A \urcorner \notin$ **S5**. Then there exists an L-model $\mathbf{M}$ such that $\|\{A, A \to \bot\}\|_{\mathbf{M}} \neq \emptyset$.*

Proof. When $\ulcorner \Diamond A \to \Box A \urcorner \notin$ **S5**, there exists a **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ and a world $w \in \mathcal{W}$ such that $\mathcal{M}, w \models \Diamond A$ and $\mathcal{M}, w \models \Diamond \neg A$. So for some $w_1 \in \mathcal{W}$: $\mathcal{M}, w_1 \models A$, and for some $w_2 \in \mathcal{W}$: $\mathcal{M}, w_2 \models \neg A$. Consider the following L-model $\mathbf{M}$:

$$\langle \{w_1, w_2\}, \mathcal{V} | \{w_1, w_2\} \rangle$$

As both A and $(A \to \bot)$ are p-wffs, it is easily seen that for the state $\{w_1, w_2\}$ of the model we have:

$$\{w_1, w_2\} \looparrowright A$$

$$\{w_1, w_2\} \looparrowright (A \to \bot)$$

Hence $\|\{A, (A \to \bot)\}\|_{\mathbf{M}} \neq \emptyset$. □

In particular, the permittance class of $\{p, p \to \bot\}$ is non-empty.

Thus the following is true:

Corollary 10. *There exist: inconsistent sets of wffs of L and L-models such that the sets have non-empty permittance classes in the models.*

Here is another example of an inconsistent set which has a non-empty permittance class.

Example 3. The set $\{p \to q, p, \neg q\}$ is inconsistent, but not plainly inconsistent. Let $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ be an L-model such that for some $w_1, w_2 \in \mathcal{W}$:

- $\mathcal{V}(p, w_1) = \mathbf{0}$,
- $\mathcal{V}(q, w_1) = \mathbf{0}$,
- $\mathcal{V}(p, w_2) = \mathbf{1}$,
- $\mathcal{V}(q, w_2) = \mathbf{0}$.

Clearly we have:

- $\mathbf{M}, w_2 \models p$ and hence $\{w_1, w_2\} \looparrowright p$,
- $\mathbf{M}, w_1 \models (p \to q)$ and thus $\{w_1, w_2\} \looparrowright (p \to q)$,
- $\mathbf{M}, w_1 \models \neg q$ as well as $\mathbf{M}, w_2 \models \neg q$; therefore $\{w_1, w_2\} \looparrowright \neg q$.

Thus $\|\{p \to q, p, \neg q\}\|_{\mathbf{M}} \neq \emptyset$.

4. Transmission of permittance

4.1. Definition and basic properties

Let us now introduce:

Definition 11 (Transmission of permittance). $X \hookrightarrow_L A$ *iff for each L-model* $\mathbf{M}$ *and each* $\mathbf{M}$*-state* σ*:*

$$\textit{if } \sigma \in \|X\|_{\mathbf{M}}, \textit{ then } \sigma \in \|A\|_{\mathbf{M}}.$$

The intuitive content of the above concept is: if each element of X is permitted by a state, then A is permitted by the state. This condition is supposed to hold for each L-model and each state of the model.

Let "$\sigma \looparrowright X$" abbreviate "for each $B \in X : \sigma \looparrowright B$".

Corollary 11. $X \hookrightarrow_L A$ *iff the following condition:*

$$\text{if } \sigma \looparrowright X, \text{ then } \sigma \looparrowright A \tag{25}$$

is fulfilled by each state σ of any L-model.

$\hookrightarrow_L$ is a consequence relation. One can easily prove:

Corollary 12. $\hookrightarrow_L$ *has the following properties:*

(Overlap): *If $A \in X$, then $X \hookrightarrow_L A$.*

(Dilution): *If $X \hookrightarrow_L A$ and $X \subseteq Y$, then $Y \hookrightarrow_L A$.*

(Cut for sets): *If $X \cup Y \hookrightarrow_L A$ and $X \hookrightarrow_L B$ for every $B \in Y$, then $X \hookrightarrow_L A$.*

$\hookrightarrow_L$ is not structural, however. The following examples illustrate this:[5]

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[5]For brevity, we use, here and below, object-level language expressions instead of their metalinguistic names.

Example 4.

$$\{\neg(p \wedge \neg q), p\} \hookrightarrow_L q \tag{26}$$

To prove (26) suppose that for some state σ of an L-model $\mathbf{M}$ it holds that:

(1) $\sigma \looparrowright \neg(p \wedge \neg q)$, and
(2) $\sigma \looparrowright p$.

By (2) there exists $w \in \sigma$, say, w_1, such that $\mathbf{M}, w_1 \models p$. But since (1) holds as well, we have $\mathbf{M}, w_1 \models \neg(p \wedge \neg q)$ and hence $\mathbf{M}, w_1 \models q$. Thus $\sigma \looparrowright q$.

Example 5.

$$\{\neg(p \wedge \neg\neg q), p\} \not\hookrightarrow_L \neg q \tag{27}$$

To see this it suffices to consider an L-model $\mathbf{M} = \langle \{w_1, w_2\}, \mathcal{V}\rangle$ in which $\mathcal{V}(p, w_1) = \mathbf{1}$, $\mathcal{V}(q, w_1) = \mathbf{0}$, $\mathcal{V}(p, w_2) = \mathbf{0}$, and $\mathcal{V}(q, w_2) = \mathbf{1}$. We get:

- $\mathbf{M}, w_1 \models p$,
- $\mathbf{M}, w_1 \models \neg(p \wedge \neg\neg q))$,
- $\mathbf{M}, w_2 \models \neg(p \wedge \neg\neg q))$

Thus $\{w_1, w_2\} \looparrowright \{\neg(p \wedge \neg\neg q), p\}$. On the other hand, since $\mathbf{M}, w_1 \not\models \neg q$, we have $\{w_1, w_2\} \not\looparrowright \neg q$.

Generally speaking, $\hookrightarrow_L$ is not structural because substitution can change the categories of wffs, that is, can turn p-wffs into n-wffs, or n-wffs into p-wffs.[6]

4.2. Transmission of permittance vs. entailment

Entailment in L, $\models_L$, can be defined by:

Definition 12 (Entailment in L). *$X \models_L A$ iff for each L-model $\mathbf{M}$:*

$$\bigcap_{B \in X} |B|_{\mathbf{M}} \subseteq |A|_{\mathbf{M}}$$

Entailment in L amounts to entailment determined by Classical Propositional Logic.

Transmission of permittance is a special case of entailment. By Corollary 6 we get:

Corollary 13. *If $X \hookrightarrow_L A$, then $X \models_L A$.*

Hence $\hookrightarrow_L$ is a *truth- preserving* consequence relation.

The converse of Corollary 13 does not hold. The following examples illustrate this:

Example 6.

$$\neg p \vee \neg q \not\hookrightarrow_L \neg(p \wedge q) \tag{28}$$

For, consider an L-model $\mathbf{M} = \langle \{w_1, w_2\}, \mathcal{V}\rangle$ such that $\mathcal{V}(p, w_1) = \mathbf{0}$, $\mathcal{V}(p, w_2) = \mathbf{1}$, and $\mathcal{V}(q, w_2) = \mathbf{1}$. Since $\neg p \vee \neg q$ is a p-wff, $\{w_1, w_2\} \looparrowright \neg p \vee \neg q$. On the other hand, $\neg(p \wedge q)$ is a n-wff and we have $\{w_1, w_2\} \not\looparrowright \neg(p \wedge q)$ because $\mathbf{M}, w_2 \models (p \wedge q)$.

[6] This can happen when the wff being substituted is a propositional variable or has the form $\neg \ldots \neg p$, where p is a propositional variable.

Example 7.

$$\{p \to q, \neg q\} \not\hookrightarrow_L \neg p \tag{29}$$

To see this it suffices to consider an L-model $\mathbf{M} = \langle \{w_1, w_2\}, \mathcal{V}\rangle$ in which $\mathcal{V}(p, w_1) = \mathbf{0}$, $\mathcal{V}(q, w_1) = \mathbf{0}$, $\mathcal{V}(p, w_2) = \mathbf{1}$, and $\mathcal{V}(q, w_2) = \mathbf{0}$. Since $\mathbf{M}, w_1 \models (p \to q)$, we get $\{w_1, w_2\} \looparrowright (p \to q)$. Clearly, $\{w_1, w_2\} \looparrowright \neg q$. But $\{w_1, w_2\} \not\looparrowright \neg p$ because $\mathcal{V}(p, w_2) = \mathbf{1}$.

4.3. Paraconsistency

As we have shown in Section 3, some inconsistent sets have non-empty permittance classes. It follows that $\hookrightarrow_L$ is *paraconsistent* in the following sense of the word: it is not the case that for every inconsistent set X and every wff B it holds that $X \hookrightarrow_L B$.

Example 8. The set $\{p \to q, p, \neg q\}$ has a non-empty permittance class (see Example 3). Hence, in particular:

$$\{p \to q, p, \neg q\} \not\hookrightarrow_L r \tag{30}$$

Example 9. The set $\{p, p \to \bot\}$ is inconsistent, but has a non-empty permittance class. One can easily show that:

$$\{p, p \to \bot\} \not\hookrightarrow_L q \tag{31}$$

Observe, however, that we still have:

$$\{p, \neg p\} \hookrightarrow_L q \tag{32}$$

4.4. Translation ( )*

The operation ( )* assigns to a wff of L the corresponding wff of $\mathcal{L}$. It is defined as follows:

Definition 13.

1. *If A is a p-wff, then $(A)^* = \Diamond A$.*
2. *If A is a n-wff, then $(A)^* = \Box A$.*

Let us stress that A in $\Diamond A$ or in $\Box A$ represents a wff of L. The operation ( )* is performed on A only once; the subformulas of A remain unaffected. In other words, ( )* is a kind of "surface translation" of wffs of L into wffs of $\mathcal{L}$.[7]

For convenience, we put:

$$(X)^* =_{df} \{(A)^* : A \in X\}$$

Let us now prove:

Lemma 3. *If $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V}\rangle$ is a* **S5***-model such that $\mathcal{M} \models (X)^*$, then $\mathcal{W} \looparrowright X$.*

[7]The idea of using translations into **S5** in constructing paraconsistent logics goes back to Jaśkowski (cf. [3], and [4] for an English translation). However, Jaśkowski's translation is defined recursively and enables an introduction of "discussive" connectives. The operation ( )* behaves differently.

Proof. First observe that $\mathcal{M}$ is the **S5**-model accompanied with an L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ w.r.t. the **M**-state $\mathcal{W}$.

The elements of $(X)^*$ are either of the form $\Diamond B$ or of the form $\Box B$, where $B \in X$.

If $\Diamond B \in (X)^*$, then, by Lemma 1, $\mathcal{M} \models \Diamond B$ yields $\mathcal{W} \looparrowright B$.

The case in which $\Box B \in (X)^*$ is analogous. □

The following holds:

Theorem 1. *X has a non-empty permittance class iff there exists a* **S5***-model $\mathcal{M}$ such that $\mathcal{M} \models (X)^*$.*

Proof. ($\Rightarrow$). Let **M** be an L-model for which $\|X\|_{\mathbf{M}} \neq \emptyset$. Let $\sigma \in \|X\|_{\mathbf{M}}$. We consider the **S5**-model $\mathcal{M}_\sigma$ accompanied with **M** w.r.t. σ, and we apply Lemma 1.
($\Leftarrow$). By Lemma 3. □

Example 10. As we have shown (see Example 3), the inconsistent set $\{p \to q, p, \neg q\}$ has a non-empty permittance class. The following takes place on the modal side:

$$\mathcal{M}_{\{w_1,w_2\}} \models \{\Diamond(p \to q), \Diamond p, \Box\neg q\} \tag{33}$$

where $\mathcal{M}_{\{w_1,w_2\}}$ is the **S5**-model accompanied (w.r.t. state $\{w_1, w_2\}$) with the L-model considered in Example 3.

However, the following holds:

Corollary 14. *If X is inconsistent and each element of X is a n-wff, then the permittance class of X is empty.*

Proof. Suppose that the permittance class of X is non-empty. Then, by Theorem 1, for some **S5**-model $\mathcal{M}$ we have $\mathcal{M} \models (X)^*$. But the elements of $(X)^*$ are of the form $\Box A$, where $A \in X$. Since $\mathcal{W}$ is non-empty, there exists a world w of $\mathcal{M}$ such that $\mathcal{M}, w \models X$. It follows that X is consistent. □

The situation can be different when X contains some p-wffs.

4.5. Transmission of permittance vs. global S5-entailment

Recall that Φ stands for a set of wffs of $\mathcal{L}$ (i.e. the modal extension of L), and ϕ is a metalanguage variable for wffs of $\mathcal{L}$.

Let us introduce:

Definition 14 (Global S5-entailment). *$\Phi \models_{\mathbf{S5}} \phi$ iff for each* **S5***-model $\mathcal{M}$: if $\mathcal{M} \models \Phi$, then $\mathcal{M} \models \phi$.*

We will now prove:

Theorem 2 (Reduction modulo $(\ )^*$). *$X \hookrightarrow_L A$ iff $(X)^* \models_{\mathbf{S5}} (A)^*$*

Proof. Suppose that $X \hookrightarrow_L A$, but $(X)^* \not\models_{\mathbf{S5}} (A)^*$. Thus for some **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ we have $\mathcal{M} \models (X)^*$ and $\mathcal{M} \not\models (A)^*$. But $\mathcal{M}$ is accompanied with the L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ w.r.t. $\mathcal{W}$, that is, $\mathcal{M} = \mathcal{M}_{\mathcal{W}}$. By Lemma 3 we get $\mathcal{W} \looparrowright X$ and hence, due to the transmission of permittance, $\mathcal{W} \looparrowright A$. If A is a p-wff, then, by Lemma 1, $\mathcal{M} \models \Diamond A$, that is, $\mathcal{M} \models (A)^*$. A contradiction. Similarly, if A is a n-wff, by Lemma 1 we get $\mathcal{M} \models \Box A$, i.e. $\mathcal{M} \models (A)^*$. A contradiction again.

Now suppose that $(X)^* \models_{\mathbf{S5}} (A)^*$, but $X \not\hookrightarrow_L A$. Then there exists a state σ of a certain L-model $\mathbf{M}$ such that $\sigma \looparrowright X$ and $\sigma \not\looparrowright A$. We consider the **S5**-model $\mathcal{M}_\sigma$ accompanied with $\mathbf{M}$ w.r.t. σ. By Lemma 1 we get $\mathcal{M}_\sigma \models (X)^*$ and $\mathcal{M}_\sigma \not\models (A)^*$. A contradiction. □

According to Theorem 2, transmission of permittance amounts to (global) **S5**-entailment among the relevant *-wffs. This does not mean that transmission of permittance can be *identified with* global **S5**-entailment. Recall that the *-wffs are either of the form $\Box A$ or of the form $\Diamond A$, where A is a wff of the non-modal language L (and thus does not involve modal operators).

Remark 5. Necessity and possibility are, in a sense, expressible in L (cf. section 2.2). But when we have $\phi \models_{\mathbf{S5}} \psi$ for $\mathcal{L}$-wffs ϕ, ψ which are of neither of the forms: $\Box A$, $\Diamond A$, the systematic replacement in ϕ and ψ of $\Box A$ by $\neg(A \to \bot)$ as well as of $\Diamond A$ by $(\neg A \to \bot)$ need not turn **S5**-entailment between ϕ and ψ into the transmission of permittance between the resultant wffs of L. For example, we have:

$$\neg\Box p \models_{\mathbf{S5}} \Box\neg\Box p \tag{34}$$

By the systematic replacement we get:

$$\neg\neg(p \to \bot) \hookrightarrow_L \neg(\neg\neg(p \to \bot) \to \bot) \tag{35}$$

(35) *does not* hold, however. To see this let us take an L-model $\mathbf{M}^* = \langle \{w_1, w_2\}, \mathcal{V} \rangle$ such that $\mathcal{V}(p, w_1) = \mathbf{0}$ and $\mathcal{V}(p, w_2) = \mathbf{1}$. Clearly, we have:

$$\{w_1, w_2\} \looparrowright \neg\neg(p \to \bot) \tag{36}$$

since $\mathbf{M}^*, w_1 \models \neg\neg(p \to \bot)$. At the same time we have:

$$\{w_1, w_2\} \not\looparrowright \neg(\neg\neg(p \to \bot) \to \bot) \tag{37}$$

because $\mathbf{M}^*, w_2 \not\models \neg(\neg\neg(p \to \bot) \to \bot)$.

To sum up: Theorem 2 does not reduce the "logic of permittance" to **S5**, but shows that one can "calculate" transmission of permittance by well-known means.

4.6. What is retained and what is lost

4.6.1. The case of single wffs. Let us first prove:

Lemma 4. *If $B \models_L A$ and (a) B and A are p-wffs, or (b) B and A are n-wffs, or (c) B is a n-wff and A is a p-wff, then $B \hookrightarrow_L A$.*

Proof. If $B \models_L A$, then $\models_L (B \to A)$ and hence $\ulcorner \Box(B \to A) \urcorner \in \mathbf{S5}$.

Assume that B and A are p-wffs. Suppose that $\Diamond B \not\models_{\mathbf{S5}} \Diamond A$. So there exists an **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ such that $\mathcal{M} \models \Diamond B$ and $\mathcal{M} \not\models \Diamond A$. Hence $\mathcal{M}, w \not\models A$ for each $w \in \mathcal{W}$, and $\mathcal{M}, w \models B$ for some $w \in \mathcal{W}$. It follows that for some $w \in \mathcal{W}$ we have $\mathcal{M}, w \not\models (B \to A)$ and therefore $\ulcorner \Box(B \to A) \urcorner \notin \mathbf{S5}$. A contradiction. Thus $\Diamond B \models_{\mathbf{S5}} \Diamond A$ and hence, by Theorem 2, $B \hookrightarrow_L A$.

Assume that B and A are n-wffs. Suppose that $\Box B \not\models_{\mathbf{S5}} \Box A$. So for some **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ we get: $\mathcal{M}, w \models B$ for any $w \in \mathcal{W}$, and $\mathcal{M}, w \not\models A$ for some $w \in \mathcal{W}$. Thus $\ulcorner \Box(B \to A) \urcorner \notin \mathbf{S5}$. A contradiction. Therefore, by Theorem 2, $B \hookrightarrow_L A$.

Finally, assume that B is a n-wff and A is a p-wff. Suppose that $\Box B \not\models_{\mathbf{S5}} \Diamond A$. Thus, for some **S5**-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, $\mathcal{M}, w \models B$ for any $w \in \mathcal{W}$, and $\mathcal{M}, w \not\models A$ for each $w \in \mathcal{W}$. Hence $\ulcorner \Box(B \to A) \urcorner \notin \mathbf{S5}$. A contradiction again. Therefore, by Theorem 2, $B \hookrightarrow_L A$. □

Thus, for instance, the following hold:

$$p \hookrightarrow_L \neg\neg p \tag{38}$$

$$\neg\neg p \hookrightarrow_L p \tag{39}$$

$$(p \to q) \hookrightarrow_L (\neg q \to \neg p) \tag{40}$$

$$(\neg q \to \neg p) \hookrightarrow_L (p \to q) \tag{41}$$

$$p \hookrightarrow_L (q \to p) \tag{42}$$

$$(p \to q) \wedge p \hookrightarrow_L q \tag{43}$$

$$(p \vee q) \wedge \neg q \hookrightarrow_L p \tag{44}$$

$$(p \vee \neg q) \wedge q \hookrightarrow_L p \tag{45}$$

$$(p \to (q \to r)) \hookrightarrow_L ((p \to q) \to (p \to r)) \tag{46}$$

$$(p \to (q \to r)) \hookrightarrow_L (p \wedge q \to r) \tag{47}$$

$$(p \wedge q \to r) \hookrightarrow_L (p \to (q \to r)) \tag{48}$$

$$(p \to (q \to r)) \hookrightarrow_L (q \to (p \to r)) \tag{49}$$

$$((p \to q) \wedge (q \to r)) \hookrightarrow_L (p \to r) \tag{50}$$

$$\neg(p \wedge q) \hookrightarrow_L (\neg p \vee \neg q) \tag{51}$$

$$\neg(p \vee q) \hookrightarrow_L (\neg p \wedge \neg q) \tag{52}$$

$$\neg(p \wedge \neg q) \hookrightarrow_L (p \to q) \tag{53}$$

$$\neg(p \to q) \hookrightarrow_L (p \wedge \neg q) \tag{54}$$

Observe, however, that the converses of (51), (52), (53) and (54) do not hold. The counterpart of *Modus Tollendo Tollens* does not hold either, i.e.:

$$((p \to q) \wedge \neg q) \not\hookrightarrow_L \neg p \tag{55}$$

because:

$$\Diamond((p \to q) \wedge \neg q) \not\models_{\mathbf{S5}} \Box\neg p \tag{56}$$

Hence:

Corollary 15. *There are cases in which: B is a p-wff, A is a n-wff, $B \models_L A$, and $B \not\hookrightarrow_L A$.*

Yet, the following holds:

$$((p \to q) \wedge \neg q) \hookrightarrow_L \oplus\neg p \tag{57}$$

(Recall that $\oplus\neg p$ claims that $\neg p$ is epistemically possible in a state.) This can be generalized.

Corollary 16. *If $B \models_L A$, B is a p-wff and A is a n-wff, then $B \hookrightarrow_L \oplus A$.*

Proof. If $B \models_L A$, then $\ulcorner\Box(B \to A)\urcorner \in \mathbf{S5}$. Suppose that $\Diamond B \not\models_{\mathbf{S5}} \Diamond \oplus A$. So for some $\mathbf{S5}$-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V}\rangle$ there exists $w_1 \in \mathcal{W}$ such that $\mathcal{M}, w_1 \models B$ and, at the same time, $\mathcal{M}, w \not\models \oplus A$ for any $w \in \mathcal{W}$. Recall that $\oplus A =_{df} (\neg A \to \bot)$. Hence for each $w \in \mathcal{W}$ we have $\mathcal{M}, w \not\models A$. Therefore $\ulcorner\Box(B \to A)\urcorner \notin \mathbf{S5}$. A contradiction. □

4.6.2. The case of sets of wffs. The direct counterpart of *Modus Ponens* holds for $\hookrightarrow_L$ (cf. 43). But we have:[8]

$$\{p \to q, p\} \not\hookrightarrow_L q \tag{58}$$

So conjunction behaves is a non-standard way in the context of $\hookrightarrow_L$: $A_1 \wedge \ldots \wedge A_n \hookrightarrow_L B$ need not be tantamount to $\{A_1, \ldots, A_n\} \hookrightarrow_L B$. The reason is that a permittance class of a set of wffs need not be equal with the permittance class of a conjunction of all the wffs in the set.[9]

Yet, the following is true:

$$\{p \to q, \boxplus p\} \hookrightarrow_L q \tag{59}$$

Recall that $\boxplus p$ can be read "p is known in a state in question".

Here are further "negative" examples:

$$\{p, q\} \not\hookrightarrow_L (p \wedge q) \tag{60}$$

$$\{p, p \to \bot\} \not\hookrightarrow_L (p \wedge \neg p) \tag{61}$$

$$\{p \vee \neg q, q\} \not\hookrightarrow_L p \tag{62}$$

$$\{p \to q, q \to r\} \not\hookrightarrow_L (p \to r) \tag{63}$$

Observe, however, that the following hold:

$$\{\boxplus p, q\} \hookrightarrow_L (p \wedge q) \tag{64}$$

and similarly for q,

$$\{\boxplus p, \boxplus q\} \hookrightarrow_L \boxplus(p \wedge q) \tag{65}$$

[8]Since $\{\Diamond(p \to q), \Diamond p\} \not\models_{\mathbf{S5}} \Diamond q$. (43) holds because $\Diamond((p \to q) \wedge p) \models_{\mathbf{S5}} \Diamond q$.

[9]For example, take an L-model $\mathbf{M} = \langle \{w_1, w_2\}, \mathcal{V}\rangle$ such that $\mathcal{V}(p, w_1) = \mathbf{0}$, $\mathcal{V}(q, w_1) = \mathbf{0}$, $\mathcal{V}(p, w_2) = \mathbf{1}$, and $\mathcal{V}(q, w_2) = \mathbf{0}$. Clearly, $\{w_1, w_2\} \in \|(p \to q), p\|_{\mathbf{M}}$, but $\{w_1, w_2\} \notin \|(p \to q) \wedge p\|_{\mathbf{M}}$. In general, a conjunction of p-wffs carries information that the conjuncts are simultaneously true in some world(s) of a state, while the information carried by the set of conjuncts amounts to the claim that each conjunct is true in a certain world of the state. When we have a "mixed" conjunction (that is, involving both p-wff and n-wffs), the information carried by n-wffs "weakens": the consecutive conjuncts, n-wffs included, are supposed to simultaneously hold in a certain world of a state.

$$\{p \vee \neg q, \boxplus q\} \hookrightarrow_L p \tag{66}$$

$$\{\neg(\neg p \wedge q), q\} \hookrightarrow_L p \tag{67}$$

$$\{\boxplus(p \to q), \boxplus(q \to r)\} \hookrightarrow_L \boxplus(p \to r) \tag{68}$$

$$\{\neg(p \wedge \neg q), \neg(q \wedge \neg r)\} \hookrightarrow_L \neg(p \wedge \neg r) \tag{69}$$

It happens that conjunction behaves in the "standard" way in the context of $\hookrightarrow_L$ although the conjuncts belong to diverse categories, as in:

$$\{p \to q, \neg(q \wedge \neg r)\} \hookrightarrow_L (p \to r) \tag{70}$$

$$\{\neg p \to q, \neg p\} \hookrightarrow_L q \tag{71}$$

$$\{p \vee q, \neg q\} \hookrightarrow_L p \tag{72}$$

$$\{\neg p \to q, \neg q\} \hookrightarrow_L p \tag{73}$$

Let us now turn to inconsistent sets. As we have shown, $\hookrightarrow_L$ is paraconsistent. But, for instance, we still have:

$$\{p \to q, p, \neg q\} \hookrightarrow_L (\neg p \vee q) \tag{74}$$

$$\{p \to q, p, \neg q\} \hookrightarrow_L (\neg(p \to q) \vee \neg p) \tag{75}$$

$$\{p \to q, p, \neg q\} \hookrightarrow_L ((\neg(p \to q) \vee \neg p) \vee q) \tag{76}$$

$$\{r, s, (r \to p), (s \to \neg p)\} \hookrightarrow_L (p \vee \neg p) \tag{77}$$

$$\{r, s, \boxplus(r \to p), \boxplus(s \to \neg p)\} \hookrightarrow_L (\boxplus r \vee \boxplus s) \tag{78}$$

5. Question raising

5.1. Questions

Let us now augment the language L with *questions*. In order to achieve this we enrich the vocabulary of L with the following signs: {, }, ?, and the comma. The new language is labelled as $L^?$. *Declarative well-formed formulas* of $L^?$ are simply the wffs of L. Questions of $L^?$ are expressions of the language falling under the schema:

$$? \, \{A_1, \ldots, A_n\} \tag{79}$$

where $n > 1$ and $A_1, \ldots, A_n$ are nonequiform, i.e. pairwise syntactically distinct, wffs of L. An expression of the form (79) can be read:

Is it the case that A_1, or ..., or is it the case that A_n? (80)

If $? \, \{A_1, \ldots, A_n\}$ is a question, then each of the wffs $A_1, \ldots, A_n$ is called a *direct answer* to the question, and these are the only direct answers to the question. A direct answer is a *possible* answer. Moreover, it constitutes a *sufficient* answer: a direct answer is supposed to provide neither less nor more information than it is requested by the corresponding question. It is *not* assumed that a direct answer must be true.[10]

[10]For details of this approach to propositional questions of formal languages see, e.g., [11], Chapter 2.

We shall use Q, Q_1, … as metalanguage variables for questions. The set of direct answers to a question Q will be denoted by $\mathbf{d}Q$.

5.1.1. Soundness of a question. We do not assign truth or falsity to questions. However, we introduce the concepts of *soundness* of a question in a world of an L-model and in a state of an L-model.[11]

Definition 15 (Soundness of a question).

1. *A question Q is sound in a world w of an L-model iff at least one direct answer to Q is true in w.*
2. *A question Q is sound in a state σ of an L-model iff Q is sound in at least one world of the state σ.*

Clearly, there are questions which are not sound in some worlds of certain L-models. Similarly, there are questions which are not sound in any states of some L-models. For example, $?\ \{p, q\}$ is not sound in any state of an L-model in which for each world w of the model it holds that $\mathcal{V}(p, w) = \mathcal{V}(q, w) = \mathbf{0}$. On the other hand, $?\ \{p, \neg p\}$ is sound in each state of any L-model, and in each world of the model.

5.2. From permittance to soundness: proto-raising

Let us now define the following relation between sets of declarative formulas of $L^?$ (i.e. wffs of L) and questions of $L^?$.

Definition 16 (Proto-raising). *A set of wffs X proto-raises a question Q (in symbols: $\mathbf{R_P}(X, Q)$) iff for each L-model $\mathbf{M}$ and each $\mathbf{M}$-state σ:*

(•) *if $\sigma \in \|X\|_{\mathbf{M}}$, then $\mathbf{M}, w \models A$ for some $w \in \sigma$ and $A \in \mathbf{d}Q$.*

The underlying intuition is: if all the wffs in X are *permitted* by a state, then Q is sound in the state, that is, at least one direct answer to Q is *true* in at least one world of the state.

We have:

Lemma 5. *Let $n > 1$. $\mathbf{R_P}(X, ?\ \{A_1, \dots, A_n\})$ iff $X \hookrightarrow_L A_1 \vee \dots \vee A_n$.*

Proof.
(⇒). Suppose that $X \not\hookrightarrow_L A_1 \vee \dots \vee A_n$. So there exist an L-model $\mathbf{M}$ and an $\mathbf{M}$-state σ such that $\sigma \looparrowright X$ and $\sigma \not\looparrowright A_1 \vee \dots \vee A_n$. Since $n > 1$, $A_1 \vee \dots \vee A_n$ is a p-wff. Thus there is no $w \in \sigma$ such that $\mathbf{M}, w \models A_i$, where $1 \leq i \leq n$. Hence it is not the case that $\mathbf{R_P}(X, ?\ \{A_1, \dots, A_n\})$.
(⇐) Take an L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ and an $\mathbf{M}$-state σ. Let $\sigma \looparrowright X$. Then $\sigma \looparrowright A_1 \vee \dots \vee A_n$. Since $n > 1$, $A_1 \vee \dots \vee A_n$ is a p-wff. Thus $\mathcal{M}_\sigma \models \Diamond(A_1 \vee \dots \vee A_n)$. Hence there exists $w \in \sigma$ such that $\mathcal{M}_\sigma, w \models A_1 \vee \dots \vee A_n$. Therefore $\mathcal{M}_\sigma, w \models A_i$ for some $1 \leq i \leq n$ and thus $\mathbf{M}, w \models A_i$ for some $1 \leq i \leq n$. It follows that $\mathbf{R_P}(X, ?\ \{A_1, \dots, A_n\})$. □

Lemma 5 together with Corollary 13 yield:

[11] Cf. [9].

Corollary 17. *If* $\mathbf{R_P}(X, ?\,\{A_1, \ldots, A_n\})$*, then* $X \models_L A_1 \vee \ldots \vee A_n$*.*
Thus if X proto-raises Q, a disjunction of all the direct answers to Q is (classically) entailed by X. Hence:

Corollary 18. *Let* w *be a world of an* L*-model. If* $\mathbf{R_P}(X, Q)$ *and all the wffs of* X *are true in* w*, then* Q *is sound in* w*.*
In other words, proto-raising secures the transmission of truth into soundness (w.r.t. worlds), which, in turn, constitutes the basic criterion of adequacy of an explication of the intuitive notion "a question Q arises from a set of declaratives X" (cf. [9], [10], [11]). However, Definition 16 cannot be regarded as providing an adequate explication of the concept. Proto-raising allows for a situation which seems forbidden in view of the intuitive notion: the permittance of all the elements of X in a state transforms into knowledge of some direct answer(s) to Q in the state. But if Q arises from X, the permittance of (all the elements of) X by a state is insufficient for knowing a direct answer to Q in the state; otherwise X would resolve Q and thus Q would not arise from X.

5.3. Giving rise

The following definition can be regarded as providing an explication of the intuitive notion "a question arises from a set of declaratives". Again, we assume that X is a set of declarative formulas of $L^?$, and Q is a question of the language.

Definition 17 (Giving rise). *A set of wffs* X *gives rise to a question* Q *(in symbols:* $\mathbf{R}(X, Q)$*) iff* $\mathbf{R_P}(X, Q)$ *and for each* $A \in \mathbf{d}Q : X \not\hookrightarrow_L A$*.*
Thus X gives rise to Q just in case X proto-raises Q, but there is no transmission of permittance between X and direct answers to Q.

We have:

Corollary 19. *If* $X \not\hookrightarrow_L A$*, then* $X \not\hookrightarrow_L \boxplus A$*.*
Thus the permittance of all the elements of X by a state does not yield the knowledge of any direct answer to Q in the state. To be more precise, there is no transmission of permittance between X and formulas which express that direct answers to Q are known.

By Lemma 5 and Definition 17 we get:

Corollary 20. *Let* $Q = ?\,\{A_1, \ldots, A_n\}$*. Then* $\mathbf{R}(X, Q)$ *iff*

1. $X \hookrightarrow_L A_1 \vee \ldots \vee A_n$*, and*
2. $X \not\hookrightarrow_L A_i$ *for* $i = 1, \ldots, n$*.*

Therefore, by the Reduction Theorem (i.e. Theorem 2):

Corollary 21. $\mathbf{R}(X, ?\,\{A_1, \ldots, A_n\})$ *iff*

1. $(X)^* \models_{\mathbf{S5}} \Diamond(A_1 \vee \ldots \vee A_n)$ *and*
2. $(X)^* \not\models_{\mathbf{S5}} (A_i)^*$ *for* $i = 1, \ldots, n$*.*

Hence the following examples come with no surprise:[12]

$$\mathbf{R}(p, q, ?\,\{p \wedge q, \neg(p \wedge q)\}) \tag{81}$$

[12]For brevity, we simply list the elements of sets of wffs.

$$\mathbf{R}(\neg p \vee \neg q, ?\ \{\neg(p \wedge q), \neg(p \vee q)\}) \tag{82}$$
$$\mathbf{R}(\neg p \wedge \neg q, ?\ \{\neg(p \vee q), p \wedge q\}) \tag{83}$$
$$\mathbf{R}(p \rightarrow q, \neg q, ?\ \{p, \neg p\}) \tag{84}$$

Observe that the raised questions have direct answers which are classically entailed by the raising sets. This is not a general rule, however.

$$\mathbf{R}(p \vee \neg p, ?\ \{p, \neg p\}) \tag{85}$$
$$\mathbf{R}(p \vee q, ?\ \{p, \neg p\}) \tag{86}$$
$$\mathbf{R}(p \vee q, ?\ \{p, q\}) \tag{87}$$
$$\mathbf{R}(p \vee q, ?\ \{p \wedge q, \neg(p \wedge q)\}) \tag{88}$$
$$\mathbf{R}(p \vee q, ?\ \{p \wedge q, p \wedge \neg q, \neg p \wedge q\}) \tag{89}$$
$$\mathbf{R}(p \rightarrow q \vee r, ?\ \{p \rightarrow q, p \rightarrow r\}) \tag{90}$$
$$\mathbf{R}(p \rightarrow q \vee r, p, ?\ \{q, r\}) \tag{91}$$
$$\mathbf{R}(\neg(q \wedge r), ?\ \{\neg q, \neg r\}) \tag{92}$$
$$\mathbf{R}(p \wedge q \rightarrow r, ?\ \{p \rightarrow r, q \rightarrow r\}) \tag{93}$$
$$\mathbf{R}(p \wedge q \rightarrow r, \neg r, ?\ \{\neg p, \neg q\}) \tag{94}$$
$$\mathbf{R}((p \vee q) \vee r, ?\ \{p, q \vee r\}) \tag{95}$$
$$\mathbf{R}(p, ?\ \{\oplus\neg p, \boxplus p\}) \tag{96}$$
$$\mathbf{R}(p \rightarrow \bot, ?\ \{\oplus p, \boxplus\neg p\}) \tag{97}$$

5.4. Question raising by inconsistencies

Questions often arise from inconsistencies. The presented account of question raising does justice to that. To be more precise, we are able to model the case in which questions arise from inconsistent sets with non-empty permittance classes. The following holds:

Corollary 22. *If* $\mathbf{R}(X, Q)$*, then the permittance class of* X *is non-empty.*

Proof. By assumption, $\mathbf{d}Q \neq \emptyset$. Let $A \in \mathbf{d}Q$. If $\mathbf{R}(X, Q)$, then $X \nrightarrow_L A$, so there exist an L-model $\mathbf{M}$ and an $\mathbf{M}$-state σ such that $\sigma \looparrowright X$ as well as $\sigma \not\looparrowright A$. Therefore X has a non-empty permittance class. □

Hence plainly inconsistent sets do not give rise to (in the sense of Definition 17) any questions. Similarly, by Corollary 14, inconsistent sets which comprise only n-wffs do not give rise to questions. The case of inconsistent sets having non-empty permittance classes in different, however.

Let us start with examples. The following hold:

$$\mathbf{R}(p \rightarrow q, p, \neg q, ?\ \{\neg(p \rightarrow q), \neg p, q\}) \tag{98}$$
$$\mathbf{R}(p \rightarrow q, p, \neg q, ?\ \{\neg(p \rightarrow q), \neg p\}) \tag{99}$$
$$\mathbf{R}(p \rightarrow q, p, \neg q, ?\ \{\neg(p \rightarrow q), q\}) \tag{100}$$
$$\mathbf{R}(p \rightarrow q, p, \neg q, ?\ \{\neg p, q\}) \tag{101}$$
$$\mathbf{R}(r, s, r \rightarrow p, s \rightarrow \neg p, ?\ \{p, \neg p\}) \tag{102}$$

$$\mathbf{R}(r, s, \boxplus(r \to p), \boxplus(s \to \neg p), ?\ \{\boxplus r, \boxplus s\}) \tag{103}$$

Let us now introduce:

Definition 18 (Complement).

1. *If A is of the form $\neg C$, then $\overline{A}$ is C.*
2. *If A is not of the form $\neg C$, then $\overline{A}$ is $\neg A$.*

Recall that, by Theorem 1, X has a non-empty permittance class iff $(X)^*$ has a **S5**-model.

Theorem 3. *If $\{A_1, \ldots, A_n\}$, where $n > 1$, is inconsistent, but has a non-empty permittance class, then $\mathbf{R}(\{A_1, \ldots, A_n\}, ?\ \{\overline{A_1}, \ldots, \overline{A_n}\})$.*

Proof. If $\{A_1, \ldots, A_n\}$ is inconsistent, then for each L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ and each $w \in \mathcal{W}$ we have $\mathbf{M}, w \not\models \{A_1, \ldots, A_n\}$ and hence $\mathbf{M}, w \models \overline{A_1} \vee \ldots \vee \overline{A_n}$. Suppose that $\mathbf{R_P}(\{A_1, \ldots, A_n\}, ?\ \{\overline{A_1}, \ldots, \overline{A_n}\})$ does not hold. Thus $\{A_1, \ldots, A_n\} \not\hookrightarrow_L \overline{A_1} \vee \ldots \vee \overline{A_n}$ and hence $(\{A_1, \ldots, A_n\})^* \not\models_{\mathbf{S5}} \Diamond(\overline{A_1} \vee \ldots \vee \overline{A_n})$. So there exists a world w of an **S5**-model $\mathcal{M}$ such that $\mathcal{M}, w \not\models \Diamond(\overline{A_1} \vee \ldots \vee \overline{A_n})$. Thus the argument of $\Diamond$ is false in each $w \in \mathcal{W}$. Therefore $\mathcal{M}, w \models \neg(\overline{A_1} \vee \ldots \vee \overline{A_n})$, that is, $\mathcal{M}, w \models \neg\overline{A_1} \wedge \ldots \wedge \neg\overline{A_n}$. But $\mathcal{M}, w \models \neg\overline{A_i}$ iff $\mathcal{M}, w \models A_i$ for $1 \leq i \leq n$. It follows that there exists a L-model $\mathbf{M}$ for which it holds that $\mathbf{M}, w \models \{A_1, \ldots, A_n\}$ and thus the analysed set is consistent. A contradiction.

Since $\{A_1, \ldots, A_n\}$ has a non-empty permittance class, there exist: an L-model $\mathbf{M}' = \langle \mathcal{W}', \mathcal{V}' \rangle$ and an $\mathbf{M}'$-state σ such that $\sigma \looparrowright A_i$ for $1 \leq i \leq n$. Suppose that $\{A_1, \ldots, A_n\} \hookrightarrow_L \overline{A_i}$ for some $1 \leq i \leq n$. Thus $\sigma \looparrowright \overline{A_i}$ and hence $\sigma \not\looparrowright A_i$. A contradiction. □

According to Theorem 3, an at least two-element finite inconsistent set of wffs gives rise to a question whose direct answers are complements of the wffs in the set – provided that the set has a non-empty permittance class. For instance:

$$\mathbf{R}(p \vee q \to r, p \wedge \neg r, ?\ \{\neg(p \vee q \to r), \neg(p \wedge \neg r)\}) \tag{104}$$

$$\mathbf{R}(p, p \to \bot, ?\ \{\neg p, \neg(p \to \bot)\}) \tag{105}$$

$$\mathbf{R}(r, s, r \to p, s \to \neg p, ?\ \{\neg r, \neg s, \neg(r \to p), \neg(s \to \neg p)\}) \tag{106}$$

The "complement" question is also raised by the empty set. In order to show this we need an auxiliary concept and two lemmas.

Definition 19. *Let Q be a question and C be a wff. By Q_C we designate a question such that $\mathbf{d}Q_C = \mathbf{d}Q \cup \{C\}$.*

When $C \notin \mathbf{d}Q$, any Q_C may be called an *extension* of Q by C.[13]

Lemma 6. *If $\mathbf{R_P}(X \cup \{B\}, Q)$, then $\mathbf{R_P}(X, Q_{\overline{B}})$.*

---

[13]Despite of their form, questions of $L^?$ are not sets of direct answers, but object-language expressions. Thus, for example, $?\ \{p, q\} \neq ?\ \{q, p\}$, although $\{p, q\} = \{q, p\}$. Hence Q_C denotes a class of expressions.

Proof. Let $Q = ?\,\{A_1, \ldots, A_n\}$ and $Q_{\overline{B}} = ?\,\{A_1, \ldots, A_n, \overline{B}\}$.

Suppose that $\mathbf{R_P}(X, ?\,\{A_1, \ldots, A_n, \overline{B}\})$ does not hold. Then, by Lemma 5, $X \not\mapsto_L A_1 \vee \ldots \vee A_n \vee \overline{B}$. Hence for some L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ and some $\mathbf{M}$-state σ we have: $\sigma \looparrowright X$ and $\sigma \not\looparrowright A_1 \vee \ldots \vee A_n \vee \overline{B}$. It follows that $\mathbf{M}, w \models \overline{A_1} \wedge \ldots \wedge \overline{A_n} \wedge B$ for each $w \in \sigma$. Therefore $\mathcal{M}_\sigma \models \Box(\overline{A_1} \wedge \ldots \wedge \overline{A_n} \wedge B)$ and hence $\mathcal{M}_\sigma \not\models \Diamond(A_1 \vee \ldots \vee A_n)$ as well as $\mathcal{M}_\sigma \models \Box B$. Thus $\mathcal{M}_\sigma \models (B)^*$ regardless of whether B is a p-wff or a n-wff, and $\mathcal{M}_\sigma \not\models (A_1 \vee \ldots \vee A_n)^*$. Since $\sigma \looparrowright X$, we have $\mathcal{M}_\sigma \models (X)^*$ and hence $\mathcal{M}_\sigma \models (X \cup \{B\})^*$. Therefore $(X \cup \{B\})^* \not\models_{\mathbf{S5}} (A_1 \vee \ldots \vee A_n)^*$. Thus, by Theorem 2, $X \cup \{B\} \not\mapsto_L A_1 \vee \ldots \vee A_n$. Since $?\,\{A_1, \ldots, A_n\}$ is a question, $n > 1$ and thus Lemma 5 applies. Hence $\mathbf{R_P}(X \cup \{B\}, ?\,\{A_1, \ldots, A_n\})$ does not hold as well.

We have assumed that $\overline{B}$ is the "last" direct answer to $Q_{\overline{B}}$. Yet, nothing essential changes when we place $\overline{B}$ at some other position. □

Lemma 7 (Deduction). *If* $\mathbf{R}(X \cup \{B\}, Q)$*, then* $\mathbf{R}(X, Q_{\overline{B}})$.

Proof. If $\mathbf{R}(X \cup \{B\}, Q)$, then $X \cup \{B\}$ has a non-empty permittance class. Let σ be an element of the class. Since $\sigma \looparrowright B$, we get $\sigma \not\looparrowright \overline{B}$ and hence $X \not\mapsto_L \overline{B}$. Clearly $X \not\mapsto_L A$ for any $A \in \mathbf{d}Q$. On the other hand, by Definition 17 and Lemma 6, $\mathbf{R}(X \cup \{B\}, Q)$ yields $\mathbf{R_P}(X, Q_{\overline{B}})$. □

Lemma 7 enables us to derive new examples from already established ones. Thus, for instance, from (101) we get:

$$\mathbf{R}(p \to q, p, ?\,\{\neg p, q\}) \tag{107}$$

while (99) gives:

$$\mathbf{R}(p \to q, \neg q, ?\,\{\neg(p \to q), \neg p\}) \tag{108}$$

From (100) we get:

$$\mathbf{R}(p \to q, p, ?\,\{\neg(p \to q), q\}) \tag{109}$$

However, the most important consequence of Lemma 7 is:

Theorem 4. *Let* $\{A_1, \ldots, A_n\}$*, where* $n > 1$*, be an inconsistent set which has a non-empty permittance class. Then* $\mathbf{R}(\emptyset, ?\,\{\overline{A_1}, \ldots, \overline{A_n}\})$.

Proof. By Theorem 3 and Lemma 7 (since $\{\overline{A_1}, \ldots, \overline{A_n}\} \cup \{\overline{A_i}\} = \{\overline{A_1}, \ldots, \overline{A_n}\}$). □

Thus, for instance:

$$\mathbf{R}(\emptyset, ?\,\{\neg(p \to q), \neg p, q\}) \tag{110}$$

$$\mathbf{R}(\emptyset, ?\,\{\neg p, \neg(p \to \bot)\}) \tag{111}$$

By the way, the following holds as well:

$$\mathbf{R}(\emptyset, ?\,\{\boxplus p, \boxplus \neg p\}) \tag{112}$$

because we have:

$$\mathbf{R}(p \to \bot, \neg p \to \bot, ?\,\{\neg(p \to \bot), \neg(\neg p \to \bot)\}) \tag{113}$$

5.5. Some comparisons

The intuitive notion "a question arises from a set of declaratives" is explicated in Inferential Erotetic Logic[14] by the concept "a set of declaratives evokes a question". Leaving aside the general schema of definition of evocation[15], in the case of the language L evocation can be defined as follows:

Definition 20 (Evocation of questions). *A set of wffs X evokes a question Q iff X entails a disjunction of all the direct answers to Q, but does not entail any single direct answer to Q.*
By "entails" we mean "entails in L"; cf. Definition 12. We write $\mathbf{E}(X, Q)$ for "X evokes Q".

Clearly we have:

Corollary 23. *Let $Q = ?\,\{A_1, \ldots, A_n\}$. Then $\mathbf{E}(X, Q)$ holds iff*

1. $X \models_L A_1 \vee \ldots \vee A_n$, *and*
2. $X \not\models_L A_i$ *for* $i = 1, \ldots, n$.

For examples of evocation see, e.g., [9], [10], [11].

Since no direct answer to an evoked question is (classically) entailed by the evoking set, we get:

Corollary 24. *If $\mathbf{E}(X, Q)$, then X is consistent.*

So evocation behaves differently than giving rise understood in the sense of Definition 17; as we have shown, some inconsistent sets give rise to questions.

However, evocation can be defined in terms of giving rise. Let us introduce:

Definition 21. $\boxplus X =_{df} \{\boxplus A : A \in X\}$
Recall that $\boxplus A$ abbreviates $\neg(A \to \bot)$ and thus can be read "A is known".

Theorem 5. $\mathbf{E}(X, ?\,\{A_1, \ldots, A_n\})$ *iff* $\mathbf{R}(\boxplus X, ?\,\{A_1, \ldots, A_n\})$.

Proof. For conciseness, let us write "$A_1 \vee \ldots \vee A_n$" as "$\bigvee A_{1,n}$".
($\Rightarrow$) If $\mathbf{E}(X, ?\,\{A_1, \ldots, A_n\})$, then $X \models_L \bigvee A_{1,n}$.
Suppose that $\mathbf{R_P}(\boxplus X, ?\,\{A_1, \ldots, A_n\})$ is not the case. Thus $\boxplus X \not\looparrowright_L \bigvee A_{1,n}$. So there exists an L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ such that for some $\mathbf{M}$-state σ: $\sigma \looparrowright \boxplus X$ and $\sigma \not\looparrowright \bigvee A_{1,n}$. Since $\bigvee A_{1,n}$ is a p-wff, it follows that $\mathbf{M}, w \not\models \bigvee A_{1,n}$ for any $w \in \mathcal{W}$. The elements of $\boxplus X$ are n-wffs of the form $\neg(B \to \bot)$. Hence $\mathbf{M}, w \models X$ for any $w \in \sigma$. Thus $X \not\models_L \bigvee A_{1,n}$. A contradiction.

[14]Generally speaking, Inferential Erotetic Logic (IEL for short) is a logic that analyses inferences in which questions play the role of conclusions and proposes criteria of validity for these inferences. For IEL see e.g. [9], [10], [11].

[15]Formulated in terms of multiple-conclusion entailment (mc-entailment for short): a set of wffs X evokes a question Q iff X mc-entails the set of direct answers to Q, but does not mc-entail any singleton set whose element is a direct answer to Q. The concept of mc-entailment generalizes the concept of entailment. Mc-entailment is a relation between *sets* of wffs. Roughly, X mc-entails Y iff the truth of all the wffs in X warrants the existence of a true wff in Y. For mc-entailment see [8].

Since $\mathbf{E}(X, ?\{A_1, \ldots, A_n\})$, then $X \not\models A_i$ for $1 \leq i \leq n$. Thus for each i, where $1 \leq i \leq n$, there exists an L-model $\mathbf{M} = \langle \mathcal{W}, \mathcal{V} \rangle$ such that for some $w \in \mathcal{W}$: $\mathbf{M}, w \models X$ and $\mathbf{M}, w \not\models A_i$. Hence $\{w\} \looparrowright \boxplus X$ and $\{w\} \not\looparrowright A_i$, that is, $\boxplus X \not\looparrowright_L A_i$.

Therefore $\mathbf{R}(\boxplus X, ?\{A_1, \ldots, A_n\})$.

($\Leftarrow$). Assume that $\mathbf{R}(\boxplus X, ?\{A_1, \ldots, A_n\})$. Hence $\boxplus X \hookrightarrow_L \bigvee A_{1,n}$. Suppose that $X \not\models_L \bigvee A_{1,n}$. So there exists a singleton state, $\{w\}$ of a certain L-model such that $\{w\} \looparrowright \boxplus X$ and $\{w\} \not\looparrowright \bigvee A_{1,n}$. Thus $\boxplus X \not\looparrowright_L \bigvee A_{1,n}$. A contradiction.

Since $\mathbf{R}(\boxplus X, ?\{A_1, \ldots, A_n\})$, we have $\boxplus X \not\looparrowright_L A_i$ for $1 \leq i \leq n$. Thus for each A_i, where $1 \leq i \leq n$, there exists a state σ_i of an L-model $\mathbf{M}$ such that $\sigma_i \looparrowright \boxplus X$ and $\sigma_i \not\looparrowright A_i$. Recall that states are, by definition, non-empty sets. Suppose that A_i is a p-wff. Hence for any $w \in \sigma_i$ we have $\mathbf{M}, w \models X$ and, at the same time, $\mathbf{M}, w \not\models A_i$. Thus $X \not\models_L A_i$ for $1 \leq i \leq n$. Now suppose that A_i is a n-wff. Since $\sigma_i \not\looparrowright A_i$, we get $\sigma_i \looparrowright \neg A_i$, where $\neg A_i$ is a p-wff. Hence for some $w \in \sigma_i$ we have $\mathbf{M}, w \not\models A_i$ and thus $X \not\models_L A_i$.

Therefore $\mathbf{E}(X, ?\{A_1, \ldots, A_n\})$.

□

Thus, generally speaking, evocation is just giving rise by premises supposed to be known. This explains why Corollary 24 holds.

6. Final remarks

Since *Ex Falso Quodlibet* holds in Classical Logic, in order to model the phenomenon of the arising of questions from inconsistencies we have to use some non-classical tools. The concept of permittance analysed in this paper is useful in this respect, although the solution offered is not fully general. An advantage of the solution lies in staying closer to the standard logical format than the alternative solutions proposed within the adaptive logic programme (see [5], [6]).

Besides its applicability in the area of questions, the concept of permittance seems interesting on its own. As we pointed out in Section 2.3, one can express the fact that A is known in a state directly in a non-modal language. The relativization to states, in turn, seems to resolve the old philosophical problem: one can legitimately claim that A is an item of knowledge in some initial state and ceases to constitute knowledge as the initial state is enriched with a new possible world/ a new account of how things are in which A is not true anymore.[16] Moreover, let us consider the case of conflicting hypotheses being general statements of the form $\forall x_i \mathcal{A}$. Assuming that they are treated semantically as we have treated p-wffs, conflicting hypotheses can be simultaneously permitted by a state and this is not tantamount to falling into a contradiction. A hypothesis of this kind constitutes an item of knowledge in a state if it is true in *each* world of the state, and extending the state with a new world in which the claim of the hypothesis does not hold only changes its epistemic status, but does not require the rejection of the hypothesis:

[16] More precisely, A ceases to constitute knowledge with respect to the "new" state.

it remains an item of knowledge in the "old" state and becomes (only) permitted in the "new" state. Permitted counterparts of n-wffs, in turn, perform the role of *state-constraints*, since in their case permittance by a state equals being true in each world of the state.

Last but not least: $\hookrightarrow_L$ seems to be an interesting truth-preserving paraconsistent consequence relation and the logic determined by it is worth further study.

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