On Search for Law-Like Statements as Abductive Hypotheses by Socratic Transformations*

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Abstract

We define a mechanism by which abductive hypotheses having the form of law-like statements are generated. We use the Socratic transformations approach as the underlying proof method.
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1 Aims

If, as Jaakko Hintikka (2007, p. 38) claims, abduction constitutes the central problem in contemporary epistemology, then designing an adequate logic of abduction is one of the most important challenges faced by contemporary logic. The logical structure of the well-known Peircean scheme of abductive reasoning is this: from an observation that A (an abductive goal), and from the known rule that if H, then A, infer H (an abductive hypothesis, or an abducible; cf (Peirce, 1958, 5.189)). However, this schema may be elaborated in detail in different ways, which lead to different models of abduction (see Urbański (2016)).

Slightly expanding the Peircean scheme, we may claim that the aim of abductive reasoning is to fill, by means of a hypothesis H, a certain gap between some dataset X (a database, a belief set, a body of knowledge) and a goal A, unattainable from X. Let us stress that both abductive hypotheses and goals may be, depending on the type of abductive reasoning, propositions, laws, rules, or even theories (cf. Gabbay and Woods (2005); Magnani (2004, 2009)). One important issue in research on abduction is whether filling this gap is intrinsically of explanatory character or not. If so, then abduction is, as a matter of

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fact, a version of the Inference to the Best Explanation (IBE), understood in the sense of Harman (1965) or Lipton (2004), or according to some refined accounts of IBE, for example Kuipers' (2004) Inference to the Best Theory. If not, then abduction may serve explanatory as well as predictive or purely deductive, or in fact any other purposes. An example of this second stance is the algorithmic perspective, proposed by Gabbay and Woods, according to which an abductive hypothesis H "is legitimately dischargeable to the extent to which it makes it possible to prove (or compute) from a database a formula not provable (or computable) from it as it is currently structured" (Gabbay and Woods, 2005, p. 88).

We shall follow the latter point of view and focus on computational issues, however with substantial explanatory flavour. Our purpose is to find a mechanism by which one can arrive at abductive hypotheses having the form of law-like statements (LLSs for short). We shall use the Socratic transformations (ST) approach (Wiśniewski, 2004c) as a proof method on which hypotheses generation mechanism will be based.

Our aim is not trivial. On the one hand, approaches to abduction based on different proof methods do not produce LLSs as outcomes; examples include Analytic Tableaux method (Aliseda (1997, 2006)), sequent calculi (Mayer and Pirri (1993)), dynamic proof method of adaptive logics (Meheus and Batens (2006); Meheus et al. (2002); it should be noted that Gauderis and Van de Putte (2012) offer account on abduction of generalizations within the adaptive logic framework). The same holds for the approaches based on the ST method proposed so far (see Urbański (2003), and Wiśniewski (2004b)). On the other hand, even though we agree that there is more to abduction than just IBE (cf. (Hintikka, 2007, p. 41–44)), there are also close affinities between abduction and search for an explanation (see Thagard (1995, 2007)). As a result, a mechanism which enables a "computation" of explanatory abductive hypotheses in the form of LLSs seems highly attractive.

In this paper we shall not consider the problem of evaluation of abductive hypotheses. This is a somewhat different issue which can be satisfactorily dealt with by computer science rather than logical means. A convincing example is offered in papers by Komosinski et al. (2014, 2012), where multi-criteria dominance relation approach is employed.

2 Socratic transformations

The ST approach offers a formal explication of the idea of solving logical problems of entailment or derivability by pure questioning, that is, by transforming the relevant initial question into consecutive questions without making any use of answers to the questions just transformed. Such *Socratic transformations* may be either successful or unsuccessful. Roughly, a successful transformation ends with a question of a specified final form, which can be answered in only one rational way. A successful transformation is a *Socratic proof.* Socratic transformations are guided by *erotetic rules*[1] which have only questions as premises and conclusions. These rules form the core of *erotetic calculi.*

We appeal here to the interrogative idea for a reason. We share Hintikka's conviction that "the interrogative approach can be argued to be a general theory

[1] "Erotetic" comes from Greek "erotema" which means "question".

of reasoning" (Hintikka et al., 1999, p. 47). Questions play far more important role in problem solving than it is typically recognized. Moreover, when explicit operations on questions, in the roles of premises or conclusions, are allowed in formal modeling of such processes, the payoff is a substantially more robust insight both into their real structure and into their computational properties. In order to justify these claims by some case-study examples we refer the reader to, int. al., Wiśniewski (2004a), Bolotov et al. (2006), Leszczyńska (2007), Urbański and Łupkowski (2010). However, although Jaakko Hintikka is nowadays one of the best-known advocates of the interrogative idea, we rely here on different assumptions and on a different approach to the logic of questions.

We shall show how the ST approach works on the example of the E^{PQ} calculus, on which our abductive mechanism will be based (see section 3). E^{PQ} is an erotetic counterpart of Pure Calculus of Quantifiers (PQ). Our presentation of this calculus will be based on the one given in Leszczyńska-Jasion et al. (2013). Detailed account on ST can be found, e. g., in Wiśniewski (2004c) and Wiśniewski and Shangin (2006). For elaboration of an erotetic background of ST, which is Inferential Erotetic Logic, see Wiśniewski (1995, 2013).

2.1 Language

Let us start with a language $\mathcal{L}$ of PQ with $\neg$ (negation), $\rightarrow$ (implication), $\wedge$ (conjunction) and $\vee$ (disjunction) as primitive connectives, and both $\forall$ (general quantifier) and $\exists$ (existential quantifier). The language $\mathcal{L}$ contains individual parameters, but it does not contain function symbols or identity. By a *term* of $\mathcal{L}$ we mean an individual variable or a parameter. We assume the usual notions of well-formed formula (wff) and sentence of $\mathcal{L}$. Now, let us extend $\mathcal{L}$ with a question-forming operator ? and the sign $\vdash$. The resulting language $\mathcal{L}^*$ has two disjoint categories of meaningful expressions: *declarative well-formed formulas* (hereafter: d-wffs), and *questions*. Questions of $\mathcal{L}^*$ are based on sequences of *atomic d-wffs* of $\mathcal{L}^*$, that is, expressions of the form:

$$S \vdash A$$

where S is a finite sequence (possibly empty) of sentences of $\mathcal{L}$, and A is a sentence of $\mathcal{L}$. A *pure sentence* is a sentence of $\mathcal{L}$ with no individual parameters. Note that atomic d-wffs of $\mathcal{L}^*$ are (single-conclusioned) sequents. A *sequent* is called *pure* if it contains only pure sentences.

In what follows we will refer to atomic d-wffs of $\mathcal{L}^*$ simply as to *sequents*, yet always having in mind that only sequents with single sentences of $\mathcal{L}$ in the succedent are taken into consideration. We use Greek lower case letters ϕ, ψ, χ, ω (possibly with subscripts) as metavariables for sequents, and Greek upper case letters Φ, Ψ, Γ as variables for sequences of sequents.

A *question* of the language $\mathcal{L}^*$ is an expression of the form:

$$?\ (\Phi)$$

where Φ is a non-empty finite sequence of sequents; the terms of this sequence are called *constituents* of the question, and we say that the question is *based on* the sequence.

Some notational conventions will be useful. The following:

$$S \,'\, T$$

stands for the *concatenation* of sequences S and T of PQ-formulas. By

$$S \,'\, A$$

we refer to the concatenation of S and the one-term sequence $\langle A \rangle$, where A is a PQ-wff. The concatenation of sequences Φ and Ψ of sequents is referred to as:

$$\Phi; \Psi$$

whereas the inscription:

$$\Phi; \phi$$

denotes the concatenation of a sequence of sequents Φ and the one-term sequence $\langle \phi \rangle$, where ϕ is a sequent. Of course, the inscription:

$$\Phi; \phi; \Psi$$

refers to the concatenation of $\Phi; \phi$ and a sequence of sequents Ψ. Any of S, T, Φ, and Ψ can be empty.

Thus when $\Phi = \langle \phi_1, \ldots, \phi_n \rangle$, the corresponding question can be written as:

$$?\ (\phi_1; \ldots; \phi_n)$$

and we will proceed that way. If $\Phi = \langle \phi \rangle$, then we write the question as:

$$?\ (\phi)$$

and we say that the question is based on a single-conclusioned sequent.

A question of the form: $?\ (S_1 \vdash A_1; \ldots; S_n \vdash A_n)$ can read: "Is it the case that: A_1 is PQ-entailed by S_1 and ... and A_n is PQ-entailed by S_n?"; due to the completeness of PQ, "PQ-entailed by" can be replaced by "PQ-derivable from." (By entailment by/derivability from a sequence we mean entailment by/derivability from the set of all the terms of the sequence.) When $n = 1$, the question pertains to the claim of a single sequent.

2.2 The calculus E^{PQ}

In a Socratic transformation one transforms a question into another question. Here is the list of erotetic rules that govern the relevant transformations of questions of $\mathcal{L}^*$:

$$\mathbf{L}_\alpha : \quad \frac{?\ (\Phi; S\ '\ \alpha\ '\ T \vdash C; \Psi)}{?\ (\Phi; S\ '\ \alpha_1\ '\ \alpha_2\ '\ T \vdash C; \Psi)} \qquad \mathbf{R}_\alpha : \quad \frac{?\ (\Phi; S \vdash \alpha; \Psi)}{?\ (\Phi; S \vdash \alpha_1; S \vdash \alpha_2; \Psi)}$$

$$\mathbf{L}_\beta : \quad \frac{?\ (\Phi; S\ '\ \beta\ '\ T \vdash C; \Psi)}{?\ (\Phi; S\ '\ \beta_1\ '\ T \vdash C; S\ '\ \beta_2\ '\ T \vdash C; \Psi)} \qquad \mathbf{R}_\beta : \quad \frac{?\ (\Phi; S \vdash \beta; \Psi)}{?\ (\Phi; S\ '\ \beta_1^* \vdash \beta_2; \Psi)}$$

$$\mathbf{L}_{\neg\neg} : \quad \frac{?\ (\Phi; S\ '\ \neg\neg A\ '\ T \vdash C; \Psi)}{?\ (\Phi; S\ '\ A\ '\ T \vdash C; \Psi)} \qquad \mathbf{R}_{\neg\neg} : \quad \frac{?\ (\Phi; S \vdash \neg\neg A; \Psi)}{?\ (\Phi; S \vdash A; \Psi)}$$

$$\mathbf{L}_\forall : \quad \frac{?(\Phi; S\ '\ \forall x_i A\ '\ T \vdash B; \Psi)}{?(\Phi; S\ '\ \forall x_i A\ '\ A(x_i/\tau)\ '\ T \vdash B; \Psi)} \qquad \mathbf{R}_\forall : \quad \frac{?(\Phi; S \vdash \forall x_i A; \Psi)}{?(\Phi; S \vdash A(x_i/\tau); \Psi)}$$

($\mathbf{L}_\forall$) provided that x_i is free in A, τ is any parameter

($\mathbf{R}_\forall$) provided that x_i is free in A, and τ is a parameter which does not occur in S nor in A

$$\mathbf{L}_\exists : \quad \frac{?(\Phi; S\ '\ \exists x_i A\ '\ T \vdash B; \Psi)}{?(\Phi; S\ '\ A(x_i/\tau)\ '\ T \vdash B; \Psi)} \qquad \mathbf{R}_\exists : \quad \frac{?(\Phi; S \vdash \exists x_i A; \Psi)}{?(\Phi; S\ '\ \forall x_i \neg A \vdash A(x_i/\tau); \Psi)}$$

($\mathbf{L}_\exists$) provided that x_i is free in A, and τ is a parameter which does not occur in S, A, T, B

($\mathbf{R}_\exists$) provided that x_i is free in A, τ is any parameter

$$\mathbf{L}_\kappa : \quad \frac{?(\Phi; S\ '\ \kappa\ '\ T \vdash C; \Psi)}{?(\Phi; S\ '\ \kappa^*\ '\ T \vdash C; \Psi)} \qquad \mathbf{R}_\kappa : \quad \frac{?(\Phi; S \vdash \kappa; \Psi)}{?(\Phi; S \vdash \kappa^*; \Psi)}$$

We shall call rules $\mathbf{R}_\alpha$ and $\mathbf{L}_\beta$ *branching rules*, as the resulting "question-conclusion" has more constituents than the "question-premise". Consequently, we will call the remaining erotetic rules *non-branching rules* (in particular, the quantificational rules of E^{PQ} are non-branching). The letters "**L**" and "**R**" indicate that the appropriate rule "operates" on the left or right side of the turnstile $\vdash$. For brevity, we have used the α, β–notation. This is explained in the following table (see Smullyan (1995)):

α	α_1	α_2	β	β_1	β_2	β_1^*
$A \wedge B$	A	B	$\neg(A \vee B)$	$\neg A$	$\neg B$	A
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	A	B	$\neg A$
$\neg(A \rightarrow B)$	A	$\neg B$	$A \rightarrow B$	$\neg A$	B	A

β_1^* may be called the *complement* of β_1.

Rules $\mathbf{L}_\kappa$ and $\mathbf{R}_\kappa$ cover the cases of quantifiers in the scope of negation and dummy quantification according to the following table:

κ	κ^*
$\neg\forall x_i A$	$\exists x_i \neg A$
$\neg\exists x_i A$	$\forall x_i \neg A$
$\forall x_i A$, where x_i is not free in A	A
$\exists x_i A$, where x_i is not free in A	A

It is easily visible that the rules of E^{PQ} are designed in such a way that each constituent of the "question-conlusion" is PQ-valid if and only if each constituent of the "question-premise" is PQ-valid. On the other hand, it can be shown that each application of a rule of E^{PQ} retains validity (in the sense of Inferential Erotetic Logic) of the corresponding erotetic inference. For a justification of the above claims see Wiśniewski (2004c) and Wiśniewski and Shangin (2006).

The concept of Socratic transformation is given by the following definition:

Definition 1. *A sequence $\langle s_1, s_2, \ldots \rangle$ of questions is a* Socratic transformation of a question $?\ (S \vdash A)$ via the rules of an erotetic calculus E^{PQ} *iff the following conditions hold:*

(i) $s_1 = ?\ (S \vdash A)$;

(ii) s_i, *where* $i > 1$, *results from* s_{i-1} *by an application of an erotetic rule of* E^{PQ}.

Consider the following example (Leszczyńska-Jasion et al., 2013, p. 977) of a Socratic transformation of sequent $\vdash \exists x P(x) \vee \exists x Q(x) \to \exists x (P(x) \vee Q(x))$:

Example 1.

1.$?(\vdash \exists x P(x) \vee \exists x Q(x) \to \exists x(P(x) \vee Q(x))$	$\mathbf{R}_\beta$
2.$?(\exists x P(x) \vee \exists x Q(x) \ \vdash\ \exists x(P(x) \vee Q(x)))$	$\mathbf{R}_\beta$
3.$?(\exists x P(x) \ \vdash\ \exists x(P(x) \vee Q(x)) \ ;\ \exists x Q(x) \ \vdash\ \exists x(P(x) \vee Q(x)))$	$\mathbf{L}_\exists$
4.$?(P(a) \ \vdash\ \exists x(P(x) \vee Q(x)) \ ;\ \exists x Q(x) \ \vdash\ \exists x(P(x) \vee Q(x)))$	$\mathbf{R}_\exists$
5.$?(P(a), \forall x \neg(P(x) \vee Q(x)) \ \vdash\ P(a) \vee Q(a) \ ;\ \exists x Q(x) \ \vdash\ \exists x(P(x) \vee Q(x)))$	$\mathbf{R}_\beta$
6.$?(P(a), \forall x \neg(P(x) \vee Q(x)), \neg P(a) \ \vdash\ Q(a) \ ;\ \exists x Q(x) \ \vdash\ \exists x(P(x) \vee Q(x)))$	$\mathbf{L}_\exists$
7.$?(P(a), \forall x \neg(P(x) \vee Q(x)), \neg P(a) \ \vdash\ Q(a) \ ;\ Q(a) \ \vdash\ \exists x(P(x) \vee Q(x)))$	$\mathbf{R}_\exists$
8.$?(P(a), \forall x \neg(P(x) \vee Q(x)), \neg P(a) \ \vdash\ Q(a) \ ;\ Q(a), \forall x \neg(P(x) \vee Q(x)) \ \vdash\ P(a) \vee Q(a))$	$\mathbf{R}_\beta$
9.$?(P(a), \forall x \neg(P(x) \vee Q(x)), \neg P(a) \ \vdash\ Q(a) \ ;\ Q(a), \forall x \neg(P(x) \vee Q(x)), \neg P(a) \ \vdash\ Q(a))$	

The last question of the above sequence has an interesting property: the affirmative answer to it is, in a sense, evident, as all the constituents of this question express some basic facts about (PQ) entailment. Thus, the answer to the first question of the sequence is also affirmative: it is true that $\exists x P(x) \vee \exists x Q(x) \to \exists x(P(x) \vee Q(x))$ is entailed by the empty set, and the sequence of example 1 is not just a transformation: it is a successful transformation, that is, a proof.

Definition 2. *Let $S \vdash A$ be a pure sequent. A finite Socratic transformation $\langle Q_1, \ldots, Q_n \rangle$ of question $?\ (S \vdash A)$ via the rules of E^{PQ} is a* Socratic proof of sequent $S \vdash A$ in the calculus E^{PQ} *iff for each constituent ϕ of Q_n:*

(a) ϕ is of the form $T \,'\, B \,'\, U \ \vdash\ B$, or

(b) ϕ is of the form $T \,'\, B \,'\, U \,'\, \neg B \,'\, W \ \vdash\ C$, or

(c) ϕ is of the form $T \,'\, \neg B \,'\, U \,'\, B \,'\, W \ \vdash\ C$.

Constituents/sequents of the form (a), (b) and (c) are called successful.

In what follows by a successful (unsuccessful) Socratic transformation we will mean a Socratic transformation which is (which is not) a Socratic proof.

Calculus E^{PQ} pertains to the Pure Calculus of Quantifiers in the following sense:

Theorem 1. *Let $S \vdash A$ be a pure sequent. $S \vdash A$ is provable in E^{PQ} iff $S \vdash A$ is PQ-valid.*

The reader will find the proof in Wiśniewski and Shangin (2006).

3 A view from E^{PQ}

Now we are in a position to define an abductive mechanism which makes use of E^{PQ}. We assume that the initial question of a Socratic transformation is based on a pure sequent (i.e. a sequent which involves only parameter-free sentences). This is not required by E^{PQ} (only Socratic proofs are supposed to start with that way), but we impose this restriction for a reason.

A law-like statement (LLS) is a first-order sentence of the form:

$$\forall x_{i_1} \dots \forall x_{i_n} (A(x_{i_1}, \dots, x_{i_n}) \to B(x_{i_1}, \dots, x_{i_n}))$$

where $A(x_{i_1}, \dots, x_{i_n})$ and $B(x_{i_1}, \dots, x_{i_n})$ are parameter-free sentential functions which involve $x_{i_1}, \dots, x_{i_n}$ as the only free variables. We consider LLS's which are expressions of $\mathcal{L}$. Let $A(x_i/\tau)$ designate a sentence which results from a sentential function Ax_i (x_i is here the only free variable of A) by the replacement of (each occurrence of) variable x_i by parameter τ. According to the rules of E^{PQ}, a wff of the form $A(x_i/\tau)$ occurs in a constituent of a question of a Socratic transformation of the considered kind due to an application of any of the rules: $\mathbf{L}_\forall$, $\mathbf{R}_\forall$, $\mathbf{L}_\exists$, $\mathbf{R}_\exists$, and is always a sentence. Moreover, such a formula never occurs in an initial question (sequent) of a Socratic proof (because the initial question has to be based on a pure sequent).

We introduce the following rule of abduction:

$$\textbf{(abd)} \qquad \frac{?(\Phi; S'A(x_i/\tau)'T \vdash B(x_i/\tau); \Psi)}{?(\Phi; S'A(x_i/\tau)'T'\forall x_i(Ax_i \to Bx_i) \vdash B(x_i/\tau); \Psi)}$$

Observe that we require that τ must replace x_i both in Ax_i and in Bx_i; in other words, it is required that the appropriate sentential functions (recall that each of them occurs in a sequent in the scope of a quantifier) must share a free variable and that this variable has been replaced by τ in both cases. In general, this is not univocal, but since we are going to extend a given Socratic transformation which starts with a question based on a pure sequent, univocality is retained.

Rule **(abd)** is supposed to be applied when we have an unsuccessful constituent in the last question of a completed Socratic transformation. Of course, it is not the case that **(abd)** is always applicable; for example, **(abd)** is not applicable to the last term of the following unsuccessful Socratic transformation (in order to improve readability, from now on we highlight a formula which the rule indicated to the right operates on):

1. $?(\exists x_1 Px_1 \vdash \forall x_1 Px_1)$ $\mathbf{L}_\exists$
2. $?(P\tau_1 \vdash \forall x_1 Px_1)$ $\mathbf{R}_\forall$
3. $?(P\tau_1 \vdash P\tau_2)$

Observe that rule **(abd)**, if applicable, enables us to "compute" a LLS given that τ is the only parameter of $A(x_i/\tau)$ and $B(x_i/\tau)$ (recall that a LLS must be parameter-free). If there are more parameters involved, the situation is more complicated (see below). Of course, unlike other rules, **(abd)** does not preserve joint validity from top to bottom.

Definition 3. *By an* abductive extension *of an unsuccessful finite Socratic transformation* $\mathbf{s} = Q_1, \ldots, Q_n$ *of* $?(S \vdash A)$ *via* E^{PQ} *we mean a finite sequence of questions* $Q_1^*, \ldots, Q_n^*, Q_{n+1}^*, \ldots, Q_u^*$ *such that:*

1. $Q_i = Q_i^*$ *for* $i = 1, \ldots, n$,
2. Q_{m+1}^* *results from* Q_m^* *by* **(abd)** *for* $m = n, n+1, \ldots, u-1$,
3. *rule* **(abd)** *is applied only with respect to unsuccessful constituents,*
4. *if rule* **(abd)** *has been applied with respect to k-th constituent of m-th* $(n \leq m < u)$ *question, then rule* **(abd)** *is not applied with respect to k-th constituent of any question with an index greater than m.*

By a *proto-abducible* of an abductive extension of $\mathbf{s}$ we mean any wff introduced to a constituent of a question of $\mathbf{s}$ by means of an application of rule **(abd)**. We say that an abductive extension is *completed* if each constituent of the last question of it is either successful or involves a proto-abducible left of the turnstile.

Clause 4 of definition 3 amounts to the requirement that **(abd)** is applied only once with respect to a given unsuccessful constituent of the last question of $\mathbf{s}$ (observe that **(abd)** is not a branching rule). In the case of a completed abductive extension of $\mathbf{s}$ rule **(abd)** has been applied to each unsuccessful constituent of the last question of $\mathbf{s}$ (these constituents are rewritten to consecutive questions and are dealt with step by step).

Example 2. (space between lines indicates where the analysed unsuccessful ST ends; the proto-abducible is underlined)

1.	$?(\forall x_1 Px_1 \vdash \forall x_1 Rx_1)$	$\mathbf{R}_\forall$
2.	$?(\forall x_1 Px_1 \vdash R\tau_1)$	$\mathbf{L}_\forall$
3.	$?(\forall x_1 Px_1, P\tau_1 \vdash R\tau_1)$	**(abd)**
4.	$?(\forall x_1 Px_1, P\tau_1, \underline{\forall x_1(Px_1 \to Rx_1)} \vdash R\tau_1)$	

Observe that $\forall x_1 Px_1 \nvDash \forall x_1 Rx_1$, but $\{\forall x_1 Px_1, \forall x_1(Px_1 \to Rx_1)\} \vDash \forall x_1 Rx_1$.

Example 3.

1.	$?(\forall x_1(Px_1 \to Rx_1) \vdash \forall x_1(Px_1 \to Gx_1))$	$\mathbf{R}_\forall$
2.	$?(\forall x_1(Px_1 \to Rx_1) \vdash P\tau_1 \to G\tau_1)$	$\mathbf{R}_\to$
3.	$?(\forall x_1(Px_1 \to Rx_1), P\tau_1 \vdash G\tau_1)$	$\mathbf{L}_\forall$
4.	$?(\forall x_1(Px_1 \to Rx_1), P\tau_1 \to R\tau_1, P\tau_1 \vdash G\tau_1)$	$\mathbf{R}_\to$
5.	$?(\forall x_1(Px_1 \to Rx_1), \neg P\tau_1, P\tau_1 \vdash G\tau_1;$ $\forall x_1(Px_1 \to Rx_1), R\tau_1, P\tau_1 \vdash G\tau_1)$	**(abd)**
6.	$?(\forall x_1(Px_1 \to Rx_1), \neg P\tau_1, P\tau_1 \vdash G\tau_1;$ $\forall x_1(Px_1 \to Rx_1), R\tau_1, P\tau_1, \underline{\forall x_1(Rx_1 \to Gx_1)} \vdash G\tau_1)$	

Again, we have $\{\forall x_1(Px_1 \to Rx_1), \forall x_1(Rx_1 \to Gx_1)\} \vDash \forall x_1(Px_1 \to Gx_1)$.

The above unsuccessful transformation 1–5 of example 3 can also be extended to:

6'. $?(\forall x_1(Px_1 \to Rx_1), \neg P\tau_1, P\tau_1 \vdash G\tau_1;$
$\forall x_1(Px_1 \to Rx_1), R\tau_1, P\tau_1, \underline{\forall x_1(Px_1 \to Gx_1)} \vdash G\tau_1)$

In this case, however, the proto-abducible is trivial, that is, it is identical with the sentence which stays right to the turnstile in the initial question (sequent).

Now, observe that in both cases we can "add" the proto-abducible to the "premises" of the initial sequent and we receive a successful Socratic transformation of the question obtained in this way (see examples 4 and 5).

Example 4.

1. $?(\forall x_1 Px_1, \forall x_1(Px_1 \to Rx_1) \vdash \forall x_1 Rx_1)$ $\mathbf{R}_\forall$
2. $?(\forall x_1 Px_1, \forall x_1(Px_1 \to Rx_1) \vdash R\tau_1)$ $\mathbf{L}_\forall$
3. $?(\forall x_1 Px_1, P\tau_1, \forall x_1(Px_1 \to Rx_1) \vdash R\tau_1)$ $\mathbf{L}_\forall$
4. $?(\forall x_1 Px_1, P\tau_1, \forall x_1(Px_1 \to Rx_1), P\tau_1 \to R\tau_1 \vdash R\tau_1)$ $\mathbf{L}_\to$
5. $?(\forall x_1 Px_1, P\tau_1, \forall x_1(Px_1 \to Rx_1), \neg P\tau_1 \vdash R\tau_1;$
$\forall x_1 Px_1, P\tau_1, \forall x_1(Px_1 \to Rx_1), R\tau_1 \vdash R\tau_1)$

Example 5.

1. $?\ ?(\forall x_1(Px_1 \to Rx_1), \forall x_1(Rx_1 \to Gx_1) \vdash \forall x_1(Px_1 \to Gx_1))$ $\mathbf{R}_\forall$
2. $?\ ?(\forall x_1(Px_1 \to Rx_1), \forall x_1(Rx_1 \to Gx_1) \vdash P\tau_1 \to G\tau_1)$ $\mathbf{R}_\to$
3. $?(\forall x_1(Px_1 \to Rx_1), \forall x_1(Rx_1 \to Gx_1), P\tau_1 \vdash G\tau_1)$ $\mathbf{L}_\forall$
4. $?(\forall x_1(Px_1 \to Rx_1), P\tau_1 \to R\tau_1\ \forall x_1(Rx_1 \to Gx_1), P\tau_1 \vdash G\tau_1)$ $\mathbf{L}_\to$
5. $?(\forall x_1(Px_1 \to Rx_1), \neg P\tau_1, \forall x_1(Rx_1 \to Gx_1), P\tau_1 \vdash G\tau_1;$ $\mathbf{L}_\forall$
$\forall x_1(Px_1 \to Rx_1), R\tau_1, \forall x_1(Rx_1 \to Gx_1), P\tau_1 \vdash G\tau_1)$
6. $?(\forall x_1(Px_1 \to Rx_1), \neg P\tau_1, \forall x_1(Rx_1 \to Gx_1), P\tau_1 \vdash G\tau_1;$ $\mathbf{L}_\to$
$\forall x_1(Px_1 \to Rx_1), R\tau_1, \forall x_1(Rx_1 \to Gx_1), R\tau_1 \to G\tau_1, P\tau_1 \vdash G\tau_1)$
7. $?(\forall x_1(Px_1 \to Rx_1), \neg P\tau_1, \forall x_1(Rx_1 \to Gx_1), P\tau_1 \vdash G\tau_1;$
$\forall x_1(Px_1 \to Rx_1), R\tau_1, \forall x_1(Rx_1 \to Gx_1), \neg R\tau_1, P\tau_1 \vdash G\tau_1;$
$\forall x_1(Px_1 \to Rx_1), R\tau_1, \forall x_1(Rx_1 \to Gx_1), G\tau_1, P\tau_1 \vdash G\tau_1)$

The above observation can be generalized. The following holds:

Theorem 2. *Let $S \vdash A$ be a pure sequent,* $\mathbf{s}$ *be a finite unsuccessful Socratic transformation of $?(S \vdash A)$ via the rules of E^{PQ}, and $\mathbf{s}^*$ be a completed abductive extension of $\mathbf{s}$ such that all the proto-abducibles of $\mathbf{s}^*$ are parameter-free. Let S^* be a sequence of all the proto-abducibles of $\mathbf{s}^*$. The sequent $S'S^* \vdash A$ is provable in E^{PQ} and thus A is CL-entailed by the set made up of all the terms of the sequence $S'S^*$.*

Proof. Let us observe that we can assign to each unsuccessful constituent of the last question of $\mathbf{s}$ exactly one proto-abducible, namely that one which is introduced when rule **(abd)** is applied with respect to this constituent. To put it differently: if i-th constituent of the last question of $\mathbf{s}$ is unsuccessful, then there exists a proto-abducible which was introduced in $\mathbf{s}^*$ when rule **(abd)** was applied with respect to i-th constituent of a question of $\mathbf{s}^*$ of an index equal or greater to the index of the last question of **s** (recall that **(abd)** is a non-branching rule, and, since $\mathbf{s}^*$ is completed, each unsuccessful constituent of the last question is "dealt with" in some question of $\mathbf{s}^*$).

We take **s** and modify it as follows:

(a) we replace each sequent, $T \vdash C$, which is a constituent of a question of **s**, with the sequent $T'S^* \vdash C$; note that $S \vdash A$ transforms into a *pure* sequent $S'S^* \vdash A$. Then we proceed analogously as in **s**;

(b) we take the leftmost unsuccessful constituent of the last question of the transformation received from **s** in the above manner. Since S^* always occurs left of the turnstile, this constituent is a sequent of the form:

$$U'A(x_i/\tau)'W'\forall x_i(Ax_i \to Bx_i)'Z \vdash B(x_i/\tau)$$

Now we apply rule $L_\forall$ with respect to the above constituent and we obtain the following constituent (of the same index) in the next question:

($) $U'A(x_i/\tau)'W' < \forall x_i(Ax_i \to Bx_i), A(x_i/\tau) \to B(x_i/\tau) >' Z \vdash B(x_i/\tau)$

In the next step we apply rule $L_\to$ with respect to ($) and we obtain two "new" successful sequents at the place where ($) has occurred;

(c) we repeat the procedure described in (b) with regard to the leftmost unsuccessful constituent of the question obtained at the previous step.

It is clear that the above procedure terminates in a finite number of steps and thus produces a finite Socratic transformation of $?(S'S^* \vdash A)$. Since unsuccessful constituents are eliminated step by step, we end with a Socratic proof of $S'S^* \vdash A$. Therefore, by soundness of E^{PQ}, A is PQ-entailed by the set made up of all the terms of $S'S^*$. This completes the proof. □

In order to obtain a general scheme we need a method of extraction of LLS's from proto-abducibles which involve parameters.

Since, by definition, both parts of an LLS must share variables, for our purposes we consider the case in which all the proto-abducibles introduced by **(abd)** are of the form:

(#) $\forall x_i(A(x_i, x_{i_1}/\tau_1, \ldots, x_{i_n}/\tau_n) \to B(x_i, x_{i_1}/\tau'_1, ..., x_{i_n}/\tau'_n))$

where $x_i, x_{i_1}, \ldots, x_{i_n}$ are distinct variables, τ_i need not be distinct from τ'_i (although can be), and $\tau_1, \ldots, \tau_n$, as well as $\tau'_1, \ldots, \tau'_n$, need not be pairwise distinct. Again, (#) is univocal due to the fact that a given unsuccessful Socratic transformation is the starting point. If (#) is a proto-abducible of the considered kind and $\tau_i = \tau'_i$ for $1 \leq i \leq n$, then the following

$$\forall x_{i_1} \ldots \forall x_{i_n} \forall x_i(A(x_i, x_{i_1}, \ldots, x_{i_n}) \to B(x_i, x_{i_1}, \ldots, x_{i_n})) \quad (1)$$

is *the abducible corresponding* to (#). If, however, $\tau_i \neq \tau'_i$ for some (but not all) i, where $1 \leq i \leq n$, then the abducible corresponding to (#) falls under the schema:

$$\forall x_{j_1} \ldots \forall x_{j_k} \forall x_i(\exists x_{j_{k+1}} \ldots \exists x_{j_n} A(x_i, x_{j_1}, \ldots, x_{j_n}) \quad (2)$$

$$\to \forall x_{j_{k+1}} \dots \forall x_{j_n} B(x_i, x_{j_1}, \dots, x_{j_n}))$$

where $x_{j_k} \dots x_{j_k}$ are all the variables among $x_{i_1}, \dots, x_{i_n}$ which are replaced in (#) by the same parameters in the antecedent and the consequent, and $x_{j_{k+1}} \dots x_{j_n}$ are all the variables among $x_{i_1}, \dots, x_{i_n}$ which are replaced in (#) by distinct parameters in the antecedent and the consequent. Finally, if $\tau_i \neq \tau_i'$ for all i, where $1 \leq i \leq n$, then the abducible has the form:

$$\forall x_i(\exists x_{i_1} \dots \exists x_{i_n} A(x_i, x_{i_1}, \dots, x_{i_n}) \to \forall x_{i_1} \dots \forall x_{i_n} B(x_i, x_{i_1}, \dots, x_{i_n})) \quad (3)$$

Note that in either case the abducible involves the "original" variables which were replaced by parameters during the initial Socratic transformation. Note also that in each case the abducible constitutes an LLS.

Example 6. (for brevity, we use x for x_1, and y for x_2)

1.	$?(\forall x \exists y Pxy \vdash \exists y \forall x Pxy)$	$\mathbf{R}_\exists$
2.	$?(\forall x \exists y Pxy, \forall y \neg \forall x Pxy \vdash \forall x Px\tau_1)$	$\mathbf{L}_\forall$
3.	$?(\forall x \exists y Pxy, \forall y \neg \forall x Pxy, \neg \forall x Px\tau_1 \vdash \forall x Px\tau_1)$	$\mathbf{R}_\forall$
4.	$?(\forall x \exists y Pxy, \forall y \neg \forall x Pxy, \neg \forall x Px\tau_1 \vdash P\tau_2\tau_1)$	$\mathbf{L}_{\neg\forall}$
5.	$?(\forall x \exists y Pxy, \forall y \neg \forall x Pxy, \exists x \neg Px\tau_1 \vdash P\tau_2\tau_1)$	$\mathbf{L}_\exists$
6.	$?(\forall x \exists y Pxy, \forall y \neg \forall x Pxy, \neg P\tau_3\tau_1 \vdash P\tau_2\tau_1)$	**(abd)**
7.	$?(\forall x \exists y Pxy, \forall y \neg \forall x Pxy, \neg P\tau_3\tau_1, \underline{\forall y(\neg P\tau_3 y \to P\tau_2 y)} \vdash P\tau_2\tau_1)$	

The abducible is $\forall y(\exists x \neg Pxy \to \forall x Pxy)$. Observe that the abducible is CL-equivalent to $\forall x \forall y Pxy$.

In order to obtain a Socratic proof of

$$\forall x \exists y Pxy, \forall y(\exists x \neg Pxy \to \forall x Pxy) \vdash \exists y \forall x Pxy$$

it is sufficient to add the abducible left of the turnstile in the initial sequent of example 6, proceed as above, apply rule $L_\forall$ to the abducible w.r.t. τ_1, apply rule $L_\to$, apply rule $L_{\neg\exists}$ to $\neg\exists x \neg Px\tau_1$ just obtained, apply rule $L_\forall$ to $\forall x \neg\neg Px\tau_1$ w.r.t. τ_3, and apply rule $L_\forall$ to $\forall x Px\tau_1$ w.r.t. τ_2.

One can prove the following:

Theorem 3. *Let $S \vdash A$ be a pure sequent. Let $\mathbf{s}$ be a finite unsuccessful Socratic transformation of $?(S \vdash A)$ via the rules of E^{PQ}, and let $\mathbf{s}^*$ be a completed abductive extension of $\mathbf{s}$ such that all the proto-abducibles of $\mathbf{s}^*$ are of the form (#) specified above. Let S^{**} be a sequence of all the abducibles which correspond to the proto-abducibles of $\mathbf{s}^*$. Then the sequent $S'S^{**} \vdash A$ is provable in E^{PQ} and thus A is CL-entailed by the set made up of all the terms of the sequence $S'S^{**}$.*

Proof. If an abducible falls under the schema (1), we proceed similarly as in the proof of Theorem 2. Suppose that an abducible is of the form (2) or of the form (3). One can get from it $\neg A(x_i/\tau, x_{i_1}/\tau_1, \dots, x_{i_n}/\tau_n)$ as well as $B(x_i/\tau, x_{i_1}/\tau_1', \dots, x_{i_n}/\tau_n')$. □

An unsuccessful Socratic transformation can be abductively extended if only rule **(abd)** is applicable to the unsucessful constituent(s) of the transformation, regardless of whether entailment/derivability holds in the initial sequent. Hence a practical problem arises: at which point one should give up in applying the rules of E^{PQ} and apply rule **(abd)**? There is no general solution to this problem. A practical advice might be: if you end with a question whose unsuccessful constituent(s) involve only atomic sentences right of the turnstile, and atomic sentences as well as compound formulas of the form $\forall x_i D$ (where x_i is free in D) left of the turnstile, try to apply rule **(abd)**. When you end with a completed abductive extension, the relevant abducibles either describe prospective goals of further deductions from accessible premises/databases (if these deductions are successfully completed, a positive solution to the main problem is arrived at) or are hypotheses to be tested (if tested with a success, you know that your problem can be resolved by means of new data).

By the way, the mechanism sketched above can be applied in proof-heuristics.

4 A view from E^{APQ}

The calculus E^{APQ} ('A' stands for 'applied') differs from E^{PQ} in language: now individual constants may occur in sequents, including the sequents to be (Socratically) proven. Moreover, instead of rules $\mathbf{L}_\forall$ and $\mathbf{R}_\exists$ of E^{PQ}, we now have:

$$\mathbf{L}^A_\forall \qquad \frac{?(\Phi; S'\forall x_i A'T \vdash B; \Psi)}{?(\Phi; S'\forall x_i A' A(x_i/\xi)'T \vdash B; \Psi)}$$

provided that x_i is free in A; ξ is a parameter or an individual constant

$$\mathbf{R}^A_\exists \qquad \frac{?(\Phi; S \vdash \exists x_i A; \Psi)}{?(\Phi; S'\forall x_i \neg A \vdash A(x_i/\xi); \Psi)}$$

provided that x_i is free in A; ξ is a parameter or an individual constant

The remaining rules of E^{APQ} are those of E^{PQ}.

The practical difference is that we are now able to consider abduction of LLS's on the basis of premises in which individual constants occur (and thus we touch the problem of explanation of facts by laws). The formal mechanism of abduction is the same as in the case of E^{PQ}, however. The rule **(abd)** is not modified, so these are only the shared parameters that count.

A weakening of the rule **(abd)** in the following way:

$$\textbf{(abd')} \qquad \frac{?(\Phi; S' A(x_i/\xi)'T \vdash B(x_i/\xi); \Psi)}{?(\Phi; S' A(x_i/\xi)'T'\forall x_i(Ax_i \to Bx_i) \vdash B(x_i/\xi); \Psi)}$$

where ξ is a parameter or an individual constant

raises a formal problem, since $A(x_i/\xi)$ and $B(x_i/\xi)$ are not univocal with respect to initial premises in which individual constants occur. Moreover, philosophical generality connected with the use of parameters is lost. On the other hand, some examples are appealing (see examples 7 and 8).

Example 7.

1.	$?(Pa \vdash Ra)$	**(abd')**
2.	$?(Pa, \underline{\forall x_1(Px_1 \to Rx_1)} \vdash Ra)$	

Example 8.

1. $?(Pa \to Ra \vdash \boxed{Pa \to Ga})$ **R**$_\to$
2. $?(\boxed{Pa \to Ra}, Pa \vdash Ga)$ **L**$_\to$
3. $?(\neg Pa, Pa \vdash Ga; \boxed{Ra}, Pa \vdash \boxed{Ga})$ **(abd')**
4. $?(\neg Pa, Pa \vdash Ga; Ra, Pa, \underline{\forall x_1(Rx_1 \to Gx_1)} \vdash Ga)$

A possible solution is to restrict **(abd')** to atomic sentences which share an individual constant and are parameter-free. Now $A(\gamma)$ stands for a parameter-free atomic sentence in which individual constant γ occurs, and similarly for $B(\gamma)$. We would have **(abd)** and the following:

$$\textbf{(abd'')} \qquad \frac{?(\Phi; S' A(\gamma)' T \vdash B(\gamma); \Psi)}{?(\Phi; S' A(\gamma)' T' \forall x_i (Ax_i \to Bx_i) \vdash B(\gamma); \Psi)}$$

Example 9.

1. $?(\boxed{Pa \vee Ra} \vdash Ga)$ **L**$_\vee$
2. $?(\boxed{Pa} \vdash \boxed{Ga}; Ra \vdash Ga)$ **(abd'')**
3. $?(Pa, \underline{\forall x_1(Px_1 \to Gx_1)} \vdash Ga; \boxed{Ra} \vdash \boxed{Ga})$ **(abd'')**
4. $?(Pa, \underline{\forall x_1(Px_1 \to Gx_1)} \vdash Ga; Ra, \underline{\forall x_1(Rx_1 \to Gx_1)} \vdash Ga)$

Observe that is *not* required that the "shared" individual constant occupies the same position in A and in B (see example 10).

Example 10.

1. $\boxed{Pab} \vdash \boxed{Rca}$ **(abd'')**
2. $Pab, \underline{\forall x_1(Px_1 b \to Rcx_1)} \vdash Rca$

A generalization of **(abd'')** to the case when there are more shared individual constants is obvious. It is unclear, however, how to define abductive extensions of unsuccessful Socratic transformations, because a "mixed" case (shared parameters and shared individual constants) may arise.

5 Concluding remarks

The algorithmic perspective offers a very broad account on abductive reasoning. One may even claim that it is too generous, and this claim can be expressed in Hintikka's (2007, p. 45) terms of distinction between definitory and strategic rules of inference as follows. In the algorithmic perspective focus on effective computability of a solution to an abductive problem may lead to overrating move-by-move correctness of a reasoning, determined by the definitory rules. This, in turn, results in underestimating the role of strategic rules, constituting the essence of abduction as an ampliative reasoning (Hintikka, 2007, p. 45–52). Thus procedures defined within the algorithmic perspective may fail to meet the criteria for full-fledged abduction. In our opinion there are two possible ways of responding to such a claim. The first one would involve conceptual considerations on the very nature of abduction, which we do not pursue in this paper. The second one is of slightly functional but still legitimate character. Our purpose here was to find a mechanism by which abductive hypotheses in the form of law-like statements can be generated. Bearing in mind the distinction between abductive process and product (Aliseda, 2006, p. 32) we do not claim that this mechanism is itself abductive, that is, that we described some kind of mental logic of abduction. What we did is this: psychological adequacy apart, we characterized an effective way of computing formulas of well-defined form of law-like statements, which may play the role of abducibles in certain contexts.

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