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An Axiomatic Account of Question Evocation: The Propositional Case

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Abstract: An axiomatic system for question evocation in Classical Propositional Logic is proposed. Soundness and completeness of the system are proven.

Keywords: erotetic logic; evocation of questions; axiomatization

1. Introduction

Inferential Erotetic Logic (IEL) is a logic that analyzes inferences in which questions perform the role of conclusions and provides an account of validity of these inferences. The idea of IEL originates from the late 1980s, but IEL was developed in depth in the 1990s as an alternative to the received view in the logic of questions, which situated the structure of questions and the question-answer relationship in the center of attention, and to the Interrogative Model of Inquiry (IMI), elaborated by Jaakko Hintikka. For IEL, see, e.g., [1–3]; for IMI, see, e.g., [4–6].

The semantic relation "a set of declarative formulas *evokes* a question" plays an important role in IEL. Validity of inferences which lead from declarative premises to questions is defined in IEL in terms of question evocation. Another semantic concept, labeled *erotetic implication*, provides an IEL-based account of validity of inferences which lead from a question and possibly some declarative premise(s) to a question. For erotetic implication see, e.g., [7], or [3], Chapter 7.

The role performed in IEL by question evocation resembles that played by entailment in a logic of statements. Thus, question evocation is worth being studied, and, as a matter of fact, it has been extensively studied (*cf.*, e.g., [1], Chapter 5, and [3], Chapter 6). Given the analogy between question evocation and entailment, it seems worthwhile to build axiomatic systems whose theorems describe what questions are evoked by what sets of declarative formulas.

1.1. Question Evocation

Speaking in very general terms, a set of declarative sentences X evokes a question Q if, and only if the hypothetical truth of all the sentences in X warrants that at least one principal possible answer (PPA) to Q is true but does not warrant the truth of any particular PPA to Q. An example may be of help. Consider:

Andrew gave a talk. If so, he talked either about philosophy or about formal logic. (1)

Did Andrew talk about philosophy, or did he talk about formal logic? (2)

If (1) consists of truths, at least one of the PPAs to (2):

Andrew talked about philosophy. (3)

Andrew talked about formal logic. (4)

must be true, but it remains undecided which one of them is true.

Question evocation is definable in terms of multiple-conclusion entailment (mc-entailment); as a matter of fact, the notion of mc-entailment is one of the main conceptual tools of IEL. Mc-entailment is a relation between sets of declarative well-formed formulas, where non-singleton sets are allowed to enter the range of the relation. By and large, a set X mc-entails a set Y just in case the hypothetical truth of all the elements of X warrants the existence of at least one truth in Y. The standard concept of entailment can now be defined as a special case, namely as mc-entailment of a singleton set. Having both concepts of entailment at hand, one can distinguish *proper mc-entailment*: a set X properly mc-entails a non-empty set Y if, and only if X mc-entails Y, but no single formula in Y is entailed by X (*cf.* [8]). Question evocation can then be defined in terms of proper mc-entailment: a set of declarative well-formed formulas X evokes a question Q if, and only if the set X properly mc-entails the set of PPAs to Q.

There are some affinities between inferences that lead to evoked questions and Hintikka's precondition for asking questions, according to which a question can only be asked if its presupposition has been established. Hintikka, however, views question asking as non-inferential moves of interrogative games. He neither assumes nor denies that a legitimately asked question has no PPA that is entailed by what has been established earlier. More importantly, the transition from Hintikka's precondition to: "the set of PPAs to a question is mc-entailed" relies on some additional assumptions which, however, need not always hold (for an analysis, see [2], pp. 19–20). Note that IEL and IMI differ conceptually in many respects. Their approaches to answerhood diverge (*cf.* [9]), and the second basic concept of IEL, namely erotetic implication, has no direct counterpart in IMI. Both theories provide different, yet somehow complementary, models of problem-solving (*cf.* [10,11]). Let us add that the concept of question evocation is also closely connected with the concept of *inquisitiveness* elaborated within the basic system of Inquisitive Semantics (*cf.* [12] for a comparison and details).

1.2. The Aim

In this paper, we present an axiomatic system whose theorems describe question evocation. We coin the system $\mathsf{PMC}_{\mathbf{E}}$. The system is a variation over the recently developed axiomatic system PMC for proper mc-entailment in Classical Propositional Logic, presented in [8]. As for PMC, one operates with sequents which have finite sets of declarative well-formed formulas on both sides of the turnstile. Some theorists identify questions with sets of statements; the relevant statements are intuitively construed as PPAs. However, a radical reductionism of this kind leads to serious difficulties (*cf.* [13,14]). When questions are reduced to sets of PPAs, PMC in its current form constitutes an axiomatic system for question evocation. However, once a non-reductionistic approach to questions is accepted, in order to get an axiomatic system for question evocation, one has to modify PMC. We have to operate with erotetic sequents which have questions as the succedents. A reformulation of axioms and rules of PMC is required, and at least one primitive structural rule is needed.

No axiomatic system for question evocation operating with erotetic sequents has been known so far, though the need for such system(s) was announced long ago (*cf.* [2]).

1.3. A Historical Digression: The Logic of Questions and the Lvov–Warsaw School

Although the logic of questions did not stay in the center of attention of the Lvov–Warsaw School, a prominent representative of the School, Kazimierz Ajdukiewicz (1890–1963), made an important contribution to the field. His 1926 note [15] puts forward an idea which is now widely accepted but was a pioneering one at the time of publication. Ajdukiewicz modeled interrogative sentences as sentential functions closed by interrogative operators. A sentential function is a formula with one or more free variables; semantically, it expresses a condition which may be satisfied by some objects and not satisfied by others. On Ajdukiewicz's account, an interrogative operator delimits the semantic range of a free variable that occurs in the corresponding sentential function. For example, 'who x' delimits the range to the set of persons, 'where x' to the set of places, and so forth. Interestingly enough, polar interrogatives are analyzed by Ajdukiewicz in a similar manner. A polar interrogative falls under the schema $[?\zeta]\zeta\phi$, where ϕ is a sentence and ζ is a variable which ranges over the set of one-argument truth-functional operators.

In the late fifties and early sixties of the 20th century, the conceptual apparatus of modern formal logic began to be extensively applied in the area of questions and questioning. A Polish logician, Tadeusz Kubiński (1923–1991), played an important and influential role in the movement. As David Harrah puts it:

> [...] Kubiński made significant contributions in many areas, on various aspects of erotetic logic. He studied several varieties of question operators, definability and reducibility of operators, various types of answers, implication and equivalence between questions, and the determining and generating of questions. ([16], p. 23)

As for the logical structure of interrogatives, Kubiński shared Ajdukiewicz's view but refined and enriched it considerably. He also put forward an interesting idea of a 'system of the logic of questions.' Theorems of some such systems are supposed to describe binary relations between questions. As for syntax, one operates with formulas of the form $Q\mathfrak{R}Q^*$, where Q and Q^* are (previously defined) questions of a formal language, and $\mathfrak{R}$ refers to a semantic relation between Q and Q^*, such as, for example, equivalence, containment, equipollence, being weaker than, being stronger than, and so forth. Kubiński also considers systems whose theorems characterize which sentences are possible answers (of different kinds) to the questions analyzed. For space reasons, we will not go into details here; some systems developed by Kubiński himself are presented in his monographs [17,18]. Let us only note that Kubiński's systems are not axiomatic. They are, however, deductive systems in the sense of being closed (as Kubiński shows) under some consequence operations.

IEL focuses its attention on *inferential* semantic relations between questions and declaratives and/or questions. However, there are obvious affinities between the system $\mathsf{PMC_E}$ presented below and Kubiński's general idea of a system of the logic of questions. Theorems of $\mathsf{PMC_E}$ are erotetic sequents, that is, are of the form $X \vdash Q$, where X is a finite set of declarative well-formed formulas and Q is a question. Intuitively, a theorem of the form $X \vdash Q$ states that a question Q is evoked by a set of declarative well-formed formulas X. Unlike Kubiński's systems, however, $\mathsf{PMC_E}$ is an axiomatic system: some erotetic sequents perform the role of axioms, and rules for deriving erotetic sequents from erotetic sequents are provided.

2. Logical Preliminaries: Syntax and Semantics

2.1. Syntax

We remain at the propositional level only, and we consider the case of Classical Propositional Logic (CPL).

2.1.1. CPL-wffs

Let $\mathcal{L}$ be the language of CPL. We assume that the vocabulary of $\mathcal{L}$ comprises a countably infinite set $\mathtt{Var}$ of propositional variables, the connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$, and brackets. The set $\mathtt{Form}$ of CPL-*wffs* is the smallest set that includes $\mathtt{Var}$ and satisfies the following conditions: (1) if $A \in \mathtt{Form}$, then '$\neg A$' $\in \mathtt{Form}$; (2) if $A, B \in \mathtt{Form}$, then '$(A \otimes B)$' $\in \mathtt{Form}$, where $\otimes$ is any of the connectives: $\vee, \wedge, \rightarrow, \leftrightarrow$. We adopt the usual conventions concerning omitting brackets. We use A, B, C, D, with subscripts when needed, as metalanguage variables for CPL-wffs, and X, Y, W, Z as metalanguage variables for sets of CPL-wffs. p, q, r are exemplary elements of $\mathtt{Var}$.

2.1.2. Questions

We enrich the vocabulary of $\mathcal{L}$ with the following signs: ?, }, {, and the comma. A *question* is an expression of the form:

$$?\{A_1, \ldots, A_n\} \tag{5}$$

where $n > 1$, and $A_1, \ldots, A_n$ are pairwise syntactically distinct CPL-wffs. An expression of the form (5) satisfying the above conditions reads:

$$\textit{Is it the case that } A_1, \textit{ or } \ldots, \textit{or is it the case that } A_n? \tag{6}$$

Note that a question *is not* a CPL-wff. However, a question is an expression of an object-level language (namely, the language $\mathcal{L}$ enriched with the above-mentioned signs).

We use $Q, Q^*, \ldots$ as metalanguage variables for questions.

We define:

$$\mathbf{d}?\{A_1, \ldots, A_n\} =_{df} \{A_1, \ldots, A_n\}. \tag{7}$$

When $?\{A_1, \ldots, A_n\}$ is a question, $\mathbf{d}?\{A_1, \ldots, A_n\}$ (*i.e.*, $\{A_1, \ldots, A_n\}$) constitutes the set of *principal possible answers* (PPAs) to the question. As in IEL, the PPAs will be also called *direct answers*.

Observe that we allow for a situation in which $\mathbf{d}Q = \mathbf{d}Q^*$, but $Q \neq Q^*$. For example, $\mathbf{d}?\{p, \neg p\} = \mathbf{d}?\{\neg p, p\}$, but $?\{p, \neg p\}$ and $?\{\neg p, p\}$ are distinct questions.

A comment. Questions are formalized in different manners. No commonly accepted logical theory of questions has been developed so far (for overviews see, e.g., [13,19,20]). In this paper, we follow the semi-reductionistic approach to questions of formal languages. On this account, questions constitute a separate category of well-formed formulas and are constructed according to the following schema:

$$?\Theta \tag{8}$$

where Θ is an expression of an object-level formal language such that Θ is *equiform with* the expression of the metalanguage which, in turn, designates the set of PPAs to the question. A question Q of the form (8) of a formal language *represents* a natural-language question Q^* construed in such a way that possible just-sufficient answers to Q^* are formalized by PPAs to Q. Remark that "just-sufficient" means here: "satisfying the request of the question by providing neither less nor more information than is requested." For details, developments and a discussion on the semi-reductionistic approach, see [1], Chapter 3, and [3], Chapter 2.

2.1.3. Erotetic Sequents

From now on, we assume that an *erotetic sequent* (e-sequent for short) falls under the schema:

$$X \vdash ?\{A_1, \ldots, A_n\} \tag{9}$$

where X is a finite (possibly empty) set of CPL-wffs and $?\{A_1, \ldots, A_n\}$ is a question, that is, $n > 1$ and $A_1, \ldots, A_n$ are pairwise syntactically distinct CPL-wffs.

Some conventions. As for e-sequents, we characterize the finite sets of CPL-wffs that occur left of the turnstile by listing the elements of these sets. When $X = \emptyset$, we write $\vdash ?\{A_1, \ldots, A_n\}$. As usual, we write X, A for $X \cup \{A\}$.

The inscription "$A \in$ CPL" means: "A is a thesis of CPL," *i.e.*, is provable in CPL. The set of theses of CPL comprises classical propositional tautologies.

A *literal* is a propositional variable or the negation of a propositional variable. We say that two literals are *complementary* if one of them is the negation of the other. A *clause* is a literal or a disjunction of literals.

2.2. Semantics

Let **1** stand for truth and **0** for falsity. A CPL-*valuation* is a function $v : \texttt{Form} \mapsto \{\mathbf{1}, \mathbf{0}\}$ satisfying the following conditions: (a) $v(\neg A) = \mathbf{1}$ iff $v(A) = \mathbf{0}$; (b) $v(A \vee B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ or $v(B) = \mathbf{1}$; (c) $v(A \wedge B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ and $v(B) = \mathbf{1}$; (d) $v(A \to B) = \mathbf{1}$ iff $v(A) = \mathbf{0}$ or $v(B) = \mathbf{1}$; (e) $v(A \leftrightarrow B) = \mathbf{1}$ iff $v(A) = v(B)$. Needless to say, there are (uncountably) many CPL-valuations.

For brevity, in what follows, we omit references to CPL. Unless otherwise stated, the semantic entailment relations defined below are supposed to hold between sets of CPL-wffs, or sets of CPL-wffs and single CPL-wffs, and by valuations we mean CPL-valuations.

We define:

Definition 1 (Entailment)**.** $X \models A$ iff for each valuation v:

- if $v(B) = \mathbf{1}$ for every $B \in X$, then $v(A) = \mathbf{1}$.

Definition 2 (Mc-entailment)**.** $X \| \!\models Y$ iff for each valuation v:

- if $v(B) = \mathbf{1}$ for every $B \in X$, then $v(A) = \mathbf{1}$ for at least one $A \in Y$.

Definition 3 (Proper mc-entailment)**.** Let $Y \neq \emptyset$. $X \| \!\lhd Y$ iff $X \| \!\models Y$ and $X \not\models A$ for every $A \in Y$.

Definition 4 (Question evocation)**.** $\mathbf{E}(X, Q)$ iff $X \| \!\lhd \mathbf{d}Q$.

In the (particular) case of CPL, we have:

Corollary 5. $\mathbf{E}(X, Q)$ *iff*

1. $X \models \bigvee \mathbf{d}Q$ *and*
2. $X \not\models A$ *for each* $A \in \mathbf{d}Q$.

3. Axioms and Primitive Rules of $\mathsf{PMC}_{\mathbf{E}}$

Axioms of $\mathsf{PMC}_{\mathbf{E}}$ are e-sequents falling under the schema:

$$\vdash ?\{D_1, \ldots, D_n\} \tag{10}$$

where each D_i $(1 \leq i \leq n)$ is a clause that does not involve complementary literals and $D_1 \vee \ldots \vee D_n$ involves complementary literals.

Since an axiom is an e-sequent, $n > 1$ and the clauses $D_1, \ldots, D_n$ are supposed to be pairwise syntactically distinct.

Here are examples of axioms of $\mathsf{PMC}_{\mathbf{E}}$:

$$\vdash ?\{p, \neg p\}, \tag{11}$$

$$\vdash ?\{\neg p, p\}, \tag{12}$$

$$\vdash ?\{p \vee \neg q, q \vee \neg p\}, \tag{13}$$

$$\vdash ?\{q \vee r \vee \neg p, p \vee r \vee \neg q\}. \tag{14}$$

The (primitive) *rules* of $\mathsf{PMC}_{\mathbf{E}}$ are:

R_1: $\dfrac{X \vdash ?\{A_1, \ldots, A_n, B\} \quad X \vdash ?\{A_1, \ldots, A_n, C\}}{X \vdash ?\{A_1, \ldots, A_n, B \wedge C\}}$ provided that $(B \wedge C) \neq A_i$ for $i = 1, \ldots, n$.

R_2: $\dfrac{X \vdash ?\{A_1, \ldots, A_n, B\}}{X \vdash ?\{A_1, \ldots, A_n, C\}}$ where $(B \leftrightarrow C) \in \mathsf{CPL}$, provided that $C \neq A_i$ for $i = 1, \ldots, n$.

R_3: $\dfrac{X \vdash ?\{B \to A_1, \ldots, B \to A_n\}}{X, B \vdash ?\{A_1, \ldots, A_n\}}$

R_4: $\dfrac{X \vdash ?\{A_1, \ldots, A_n\}}{X \vdash ?\{B_1, \ldots, B_n\}}$ where $\mathbf{d}?\{A_1, \ldots, A_n\} = \mathbf{d}?\{B_1, \ldots, B_n\}$.

The provisos in rules R_1 and R_2 secure that the corresponding rules produce e-sequents (recall that direct answers to a question are supposed to be pairwise syntactically distinct). Rule R_4 is not superfluous. Recall that the semi-reductionistic approach to questions allows for the existence of distinct questions that have equal sets of PPAs. Rule R_4 enables a transition from a question to a syntactically distinct question, which, however, has the same set of PPAs.

A *proof* of an e-sequent $X \vdash Q$ in $\mathsf{PMC}_{\mathbf{E}}$ is a finite labeled tree regulated by the rules of $\mathsf{PMC}_{\mathbf{E}}$, where the leaves are labeled with axioms and the e-sequent $X \vdash Q$ labels the root. An e-sequent is *provable* in $\mathsf{PMC}_{\mathbf{E}}$ iff it has at least one proof in $\mathsf{PMC}_{\mathbf{E}}$.

Here are examples of proofs:

Example 1. $p \lor \neg p \vdash ?\{p, \neg p\}$

$$\begin{array}{cl} \vdash ?\{p, \neg p\} & (\mathsf{Ax}) \\ \vdash ?\{p, p \lor \neg p \to \neg p\} & (\mathsf{R}_2) \\ \vdash ?\{p \lor \neg p \to \neg p, p\} & (\mathsf{R}_4) \\ \vdash ?\{p \lor \neg p \to \neg p, p \lor \neg p \to p\} & (\mathsf{R}_2) \\ p \lor \neg p \vdash ?\{\neg p, p\} & (\mathsf{R}_3) \\ p \lor \neg p \vdash ?\{p, \neg p\} & (\mathsf{R}_4) \end{array}$$

Example 2. $p \lor q \vdash ?\{p, q\}$

$$\begin{array}{cl} \vdash ?\{\neg q \lor p, \neg p \lor q\} & (\mathsf{Ax}) \\ \vdash ?\{\neg q \lor p, p \lor q \to q\} & (\mathsf{R}_2) \\ \vdash ?\{p \lor q \to q, \neg q \lor p\} & (\mathsf{R}_4) \\ \vdash ?\{p \lor q \to q, p \lor q \to p\} & (\mathsf{R}_2) \\ p \lor q \vdash ?\{q, p\} & (\mathsf{R}_3) \\ p \lor q \vdash ?\{p, q\} & (\mathsf{R}_4) \end{array}$$

Example 3. $p \to q \lor r, p \vdash ?\{q, r\}$

$$\begin{array}{cl} \vdash ?\{\neg r \lor \neg p \lor q, \neg q \lor \neg p \lor r\} & (\mathsf{Ax}) \\ \vdash ?\{\neg r \lor \neg p \lor q, (p \to q \lor r) \to (p \to r)\} & (\mathsf{R}_2) \\ \vdash ?\{(p \to q \lor r) \to (p \to r), \neg r \lor \neg p \lor q)\} & (\mathsf{R}_4) \\ \vdash ?\{(p \to q \lor r) \to (p \to r), (p \to q \lor r) \to (p \to q)\} & (\mathsf{R}_2) \\ \vdash ?\{(p \to q \lor r) \to (p \to q), (p \to q \lor r) \to (p \to r)\} & (\mathsf{R}_4) \\ p \to q \lor r \vdash ?\{p \to q, p \to r\} & (\mathsf{R}_3) \\ p \to q \lor r, p \vdash ?\{q, r\} & (\mathsf{R}_3) \end{array}$$

Example 4. $p \land q \to r, \neg r \vdash ?\{\neg p, \neg q\}$

$$\begin{array}{cl} \vdash ?\{q \lor r \lor \neg p, p \lor r \lor \neg q\} & (\mathsf{Ax}) \\ \vdash ?\{q \lor r \lor \neg p, (p \land q \to r) \to (\neg r \to \neg q)\} & (\mathsf{R}_2) \\ \vdash ?\{(p \land q \to r) \to (\neg r \to \neg q), q \lor r \lor \neg p\} & (\mathsf{R}_4) \\ \vdash ?\{(p \land q \to r) \to (\neg r \to \neg q), (p \land q \to r) \to (\neg r \to \neg p)\} & (\mathsf{R}_2) \\ \vdash ?\{(p \land q \to r) \to (\neg r \to \neg p), (p \land q \to r) \to (\neg r \to \neg q)\} & (\mathsf{R}_4) \\ p \land q \to r \vdash ?\{\neg r \to \neg p, \neg r \to \neg q\} & (\mathsf{R}_3) \\ p \land q \to r, \neg r \vdash ?\{\neg p, \neg q\} & (\mathsf{R}_3) \end{array}$$

Example 5. $\vdash ?\{p, q, \neg(p \vee q)\}$

$$\dfrac{\dfrac{\vdash ?\{p,q,\neg p\} \quad (\mathsf{Ax}) \qquad \vdash ?\{p,q,\neg q\} \quad (\mathsf{Ax})}{\vdash ?\{p,q,\neg p \wedge \neg q\}} \ (\mathsf{R}_1)}{\vdash ?\{p,q,\neg(p \vee q)\}} \ (\mathsf{R}_2)$$

4. Soundness and Completeness of $\mathsf{PMC}_\mathbf{E}$

4.1. Soundness

The proof of soundness of $\mathsf{PMC}_\mathbf{E}$ is very similar to the proof of Theorem 1 in [8]. The following are true:

Proposition 6. *If $\vdash Q$ is an axiom of $\mathsf{PMC}_\mathbf{E}$, then $\mathbf{E}(\emptyset, Q)$.*

Proof. Let $\vdash ?\{D_1, \ldots, D_n\}$ be an axiom of $\mathsf{PMC}_\mathbf{E}$. Since each D_j, where $1 \leq j \leq n$, is a clause that involves no complementary literals, we have $\emptyset \not\models D_j$ for $j = 1, \ldots, n$. However, $D_1 \vee \ldots \vee D_n$ involves complementary literals and thus $\emptyset \Vdash \{D_1, \ldots, D_n\}$. Therefore, $\mathbf{E}(\emptyset, ?\{D_1, \ldots, D_n\})$. □

Proposition 7. *The rules of $\mathsf{PMC}_\mathbf{E}$ preserve question evocation from top to bottom.*

Proof. We proceed by cases.

(Rule R_1). Assume that $\mathbf{E}(X, ?\{A_1, \ldots, A_n, B\})$ and $\mathbf{E}(X, ?\{A_1, \ldots, A_n, C\})$ hold. Hence, $X \Vdash \{A_1, \ldots, A_n, B\}$ and $X \Vdash \{A_1, \ldots, A_n, C\}$. Suppose that $X \not\Vdash \{A_1, \ldots, A_n, B \wedge C\}$. Therefore, $X \not\Vdash \{A_1, \ldots, A_n, B\}$ or $X \not\Vdash \{A_1, \ldots, A_n, C\}$—a contradiction. Thus, $X \Vdash \{A_1, \ldots, A_n, B \wedge C\}$. Now suppose that $X \models B \wedge C$. Then, $X \models B$ and $X \models C$. Therefore, neither $\mathbf{E}(X, ?\{A_1, \ldots, A_n, B\})$ nor $\mathbf{E}(X, ?\{A_1, \ldots, A_n, C\})$ holds—a contradiction. Hence, $X \not\models B \wedge C$. Since $\mathbf{E}(X, ?\{A_1, \ldots, A_n, B\})$ is the case, we have $X \not\models A_i$ for $i = 1, \ldots, n$. Therefore, $\mathbf{E}(X, \{A_1, \ldots, A_n, B \wedge C\})$ holds.

(Rule R_3). Assume that $\mathbf{E}(X \cup \{B\}, ?\{A_1, \ldots, A_n\})$ does not hold. Thus, (a) $X \cup \{B\} \not\Vdash \{A_1, \ldots, A_n\}$ or (b) $X \cup \{B\} \models A_j$ for some $1 \leq j \leq n$. If (a) is the case, then $X \not\Vdash \{B \to A_1, \ldots, B \to A_n\}$ and hence $\mathbf{E}(X, ?\{B \to A_1, \ldots, B \to A_n\})$ does not hold. If (b) is the case, then $X \models B \to A_j$ for some $1 \leq j \leq n$ and, again, $\mathbf{E}(X, ?\{B \to A_1, \ldots, B \to A_n\})$ does not hold.

The cases of rules R_2 and R_4 are obvious. □

Thus, we get:

Theorem 8 (Soundness)**.** *If the e-sequent $X \vdash Q$ is provable in $\mathsf{PMC}_\mathbf{E}$, then $\mathbf{E}(X, Q)$.*

Proof. By Propositions 6 and 7. □

4.2. Completeness

The completeness proof presented below is based on similar ideas as the completeness proof of PMC given in [8]; the differences stem from the fact that one has to secure that the appropriate trees are labeled with e-sequents. Moreover, we make use of some properties of question evocation.

We say that an e-sequent with the empty antecedent, $\vdash ?\{A_1, \ldots, A_n\}$, is in *normal form* iff every $A \in \{A_1, \ldots, A_n\}$ is a conjunction of one or more clauses; by the conjunction of one clause we mean the clause itself. In other words, an e-sequent with the empty antecedent is in the normal form iff every direct answer to the question that constitutes the succedent is in the conjunctive normal form. Recall that clauses are, by definition, the simplest cases of CPL-wffs in the conjunctive normal form.

Observe that the axioms of $\mathsf{PMC}_{\mathbf{E}}$ are in the normal form.

By the *rank* of the succedent Q of an e-sequent in the normal form, we mean the number of occurrences of the conjunction connective, $\wedge$, in the CPL-wffs belonging to the set $\mathbf{d}Q$; the rank of Q is designated by $\mathsf{r}(Q)$.

Lemma 9. *Let* $\vdash Q$ *be an e-sequent in the normal form. If* $\mathbf{E}(\emptyset, Q)$*, then* $\vdash Q$ *is provable in* $\mathsf{PMC}_{\mathbf{E}}$.

Proof. We proceed by induction on the rank of Q.
1. $\mathsf{r}(Q) = 0$. In this case, each element of $\mathbf{d}Q$ is a clause, and a disjunction of all the elements of $\mathbf{d}Q$ is a clause. Assume that $\mathbf{E}(\emptyset, Q)$. Thus, no clause in $\mathbf{d}Q$ involves complementary literals (since no clause in $\mathbf{d}Q$ is valid) and a disjunction of all the clauses of $\mathbf{d}Q$ involves complementary literal(s) (because it is valid). Hence, $\vdash Q$ is an axiom of $\mathsf{PMC}_{\mathbf{E}}$ and thus is provable in $\mathsf{PMC}_{\mathbf{E}}$.

2. $\mathsf{r}(Q) > 0$. Assume that $\mathbf{E}(\emptyset, Q)$, where $\mathsf{r}(Q) > 0$. Let $Q = ?\{A_1, \ldots, A_n\}$. Thus, there exists at least one index i, where $1 \leq i \leq n$, such that A_i is of the form $B_1 \wedge \ldots \wedge B_m$, where $m > 1$ and $B_1, \ldots, B_m$ are clauses. Let j be the least index that fulfills the above condition. Let:

$$Q^* = ?\{A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n, B_1 \wedge \ldots \wedge B_m \wedge (p_t \vee \neg p_t) \wedge (p_s \vee \neg p_s)\}, \tag{15}$$

where p_t, p_s are propositional variables that do not occur in Q, and $p_t \neq p_s$.

Since $\mathbf{E}(\emptyset, Q)$, we also have $\mathbf{E}(\emptyset, Q^*)$. It follows that:

$$\emptyset \not\models B_1 \wedge \ldots \wedge B_m \wedge (p_t \vee \neg p_t) \wedge (p_s \vee \neg p_s), \tag{16}$$

and therefore:

$$\emptyset \not\models B_1 \wedge \ldots \wedge B_m. \tag{17}$$

Thus, there exists a least index, say, e, where $1 \leq e \leq m$, such that:

$$\emptyset \not\models B_e. \tag{18}$$

Let:

$$Q_1^* = ?\{A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n, B_e \wedge (p_t \vee \neg p_t)\}, \tag{19}$$

$$Q_2^* = ?\{A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n, B_1 \wedge \ldots \wedge B_{e-1} \wedge B_{e+1} \wedge \ldots \wedge B_m \wedge (p_s \vee \neg p_s)\}. \tag{20}$$

Since neither p_t nor p_s occurs in a wff that belongs to $\mathbf{d}Q$, Q_1^* and Q_2^* are questions; recall that direct answers to a question are supposed to be pairwise syntactically distinct.

Observe that $\mathsf{r}(Q_1^*) < \mathsf{r}(Q^*)$ and $\mathsf{r}(Q_2^*) < \mathsf{r}(Q^*)$. Thus, by the induction hypothesis:

(a) if $\mathbf{E}(\emptyset, Q_1^*)$, then the e-sequent $\vdash Q_1^*$ is provable in $\mathsf{PMC}_{\mathbf{E}}$,
(b) if $\mathbf{E}(\emptyset, Q_2^*)$, then the e-sequent $\vdash Q_2^*$ is provable in $\mathsf{PMC}_{\mathbf{E}}$.

Clearly, we have $\mathbf{E}(\emptyset, Q_1^*)$. Thus, by (a), $\vdash Q_1^*$ is provable in $\mathsf{PMC}_{\mathbf{E}}$.

As for Q_2^*, we have $\emptyset \models \bigvee \mathbf{d}Q_2^*$.

There are two cases to be considered.

(*Case 1.*) $\emptyset \not\models B_1 \wedge \ldots \wedge B_{e-1} \wedge B_{e+1} \wedge \ldots \wedge B_m \wedge (p_s \vee \neg p_s)$. Hence, $\mathbf{E}(\emptyset, Q_2^*)$ and therefore, by (b), the e-sequent $\vdash Q_2^*$ is provable in $\mathsf{PMC}_{\mathbf{E}}$. Since we have rule R_1 and the e-sequent $\vdash Q_1^*$ is provable as well, the e-sequent:

$$\vdash ?\{A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n, B_e \wedge (p_t \vee \neg p_t) \wedge B_1 \wedge \ldots \wedge B_{e-1} \wedge B_{e+1} \wedge \ldots \wedge B_m \wedge (p_s \vee \neg p_s)\} \tag{21}$$

is provable in the calculus. However, we also have rule R_2 and hence the e-sequent:

$$\vdash ?\{A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_k, B_1 \wedge \ldots \wedge B_{e-1} \wedge B_e \wedge B_{e+1} \wedge \ldots \wedge B_m\} \tag{22}$$

is provable as well. Recall that $A_j = B_1 \wedge \ldots \wedge B_{e-1} \wedge B_e \wedge B_{e+1} \wedge \ldots \wedge B_m$. By applying rule R_4 to the e-sequent (22), we get the e-sequent:

$$\vdash ?\{A_1, \ldots, A_n\} \tag{23}$$

that is, the e-sequent $\vdash Q$ is provable in $\mathsf{PMC}_{\mathbf{E}}$.

(*Case 2.*) $\emptyset \models B_1 \wedge \ldots \wedge B_{e-1} \wedge B_{e+1} \wedge \ldots \wedge B_m \wedge (p_s \vee \neg p_s)$. Therefore, $B_e \wedge (p_t \vee \neg p_t)$ is CPL-equivalent to A_j. As the e-sequent $\vdash Q_1^*$ is provable in $\mathsf{PMC}_{\mathbf{E}}$ and we have rule R_2, it follows that the e-sequent:

$$\vdash ?\{A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n, A_j\} \tag{24}$$

is provable in the calculus as well. By applying rule R_4 to the e-sequent (24), we get the e-sequent (23) as required. □

The following holds:

Proposition 10.
For each question $?\{A_1, \ldots . A_n\}$ there exists a question $?\{C_1, \ldots, C_n\}$ such that $C_1, \ldots, C_n$ are in the conjunctive normal form, $(A_i \leftrightarrow C_i) \in$ CPL for $i = 1, \ldots, n$, and $\{C_1, \ldots, C_n\} \cap \{A_1, \ldots, A_n\} = \emptyset$.

Proof. We define the set $\{C_1, \ldots, C_n\}$ as follows:

1. $C_1 = B \wedge (p_j \vee \neg p_j)$, where B is an arbitrary but fixed CPL-wff in the conjunctive normal form such that $(A_1 \leftrightarrow B) \in$ CPL, and p_j is a propositional variable that does not occur in A_1.
2. if $i > 1$, then $C_i = D \wedge (p_k \vee \neg p_k)$, where D is an arbitrary but fixed CPL-wff in the conjunctive normal form such that $(A_i \leftrightarrow D) \in$ CPL, and p_k is a propositional variable that occurs neither in $C_1, \ldots, C_{i-1}$ nor in $A_1, \ldots, A_n$.

□

Clearly, we also have:

Proposition 11. *If $\mathbf{E}(X, ?\{A_1, \ldots, A_n\})$ and $?\{C_1, \ldots, C_n\}$ is a question such that $C_1, \ldots, C_n$ are in the conjunctive normal form and $(A_i \leftrightarrow C_i) \in$ CPL for $i = 1, \ldots, n$, then $\mathbf{E}(X, ?\{C_1, \ldots, C_n\})$.*

Let us now prove:

Lemma 12. *If $\mathbf{E}(\emptyset, Q)$, then the e-sequent $\vdash Q$ is provable in* $\mathsf{PMC}_{\mathbf{E}}$.

Proof. Assume that $\mathbf{E}(\emptyset, Q)$. Let $Q = ?\{A_1, \ldots, A_n\}$. Let $?\{C_1, \ldots, C_n\}$ be an arbitrary but fixed question that has the properties specified by Proposition 10 w.r.t. Q. By Proposition 11, $\mathbf{E}(\emptyset, ?\{C_1, \ldots, C_n\})$. The e-sequent:

$$\vdash ?\{C_1, \ldots, C_n\} \tag{25}$$

is in the normal form. Therefore, by Lemma 9, the e-sequent (25) is provable in $\mathsf{PMC}_{\mathbf{E}}$. We can extend a proof of the e-sequent (25) as follows:

$$\begin{array}{ll}
\vdash ?\{C_1, \ldots, C_{n-1}, A_n\} & (R_2) \\
\vdash ?\{C_1, \ldots, A_n, C_{n-1}\} & (R_4) \\
\vdash ?\{C_1, \ldots, A_n, A_{n-1}\} & (R_2) \\
\ldots & \\
\vdash ?\{A_n, A_{n-1}, \ldots, C_1\} & (R_4) \\
\vdash ?\{A_n, A_{n-1}, \ldots, A_1\} & (R_2) \\
\vdash ?\{A_1, \ldots, A_n\} & (R_4).
\end{array}$$

Thus, the e-sequent $\vdash ?\{A_1, \ldots, A_n\}$, *i.e.*, $\vdash Q$, is provable in $\mathsf{PMC}_{\mathbf{E}}$. □

We also need:

Proposition 13. *Let* $X = \{B_1, \ldots, B_m\}$. $\mathbf{E}(X, ?\{A_1, \ldots, A_n\})$ *iff* $\mathbf{E}(\emptyset, ?\{B_1 \to (B_2 \to (\ldots \to (B_m \to A_1)\ldots)), \ldots, B_1 \to (B_2 \to (\ldots \to (B_m \to A_n)\ldots))\})$.

We are now ready to prove:

Theorem 14 (Completeness)**.** *Let* X *be a finite set of wffs. If* $\mathbf{E}(X, Q)$, *then the e-sequent* $X \vdash Q$ *is provable in* $\mathsf{PMC}_{\mathbf{E}}$.

Proof. Since we have already proven Lemma 12, it suffices to consider the case in which $X \neq \emptyset$.
Let $X = \{B_1, \ldots, B_m\}$ and $Q = ?\{A_1, \ldots, A_n\}$.
Assume that $\mathbf{E}(X, Q)$. Thus, by Proposition 13, we have:

$$\mathbf{E}(\emptyset, ?\{B_1 \to (B_2 \to (\ldots \to (B_m \to A_1)\ldots)), \ldots, B_1 \to (B_2 \to (\ldots \to (B_m \to A_n)\ldots))\})$$

and, therefore, by Lemma 12, the e-sequent:

$$\vdash ?\{B_1 \to (B_2 \to (\ldots \to (B_m \to A_1)\ldots)), \ldots, \qquad (26)$$

$$B_1 \to (B_2 \to (\ldots \to (B_m \to A_n)\ldots))\}$$

is provable in $\mathsf{PMC}_{\mathbf{E}}$. One can extend a proof of the e-sequent (26) as follows:

$$B_1 \vdash ?\{B_2 \to (\ldots \to (B_m \to A_1)\ldots), \ldots, B_2 \to (\ldots \to (B_m \to A_n)\ldots)\} \quad (\mathsf{R}_3)$$

$$\ldots$$

$$B_1, B_2, \ldots, B_{m-1} \vdash ?\{B_m \to A_1, \ldots, B_m \to A_n\} \qquad (\mathsf{R}_3)$$

$$B_1, \ldots, B_m \vdash ?\{A_1, \ldots, A_n\} \qquad (\mathsf{R}_3)$$

Hence, the e-sequent $X \vdash Q$ is provable in $\mathsf{PMC}_{\mathbf{E}}$. □

5. Derived Rules and Admissible Rules

5.1. Some Derived Rules

Rules R_1 and R_2 operate on the rightmost direct answers. However, due to the presence of rule R_4, one can always transform a question by putting a direct answer at the rightmost position, act upon the answer, and then move the resultant wff at the initial position of the answer acted upon. In other words, the following are derived rules of the calculus $\mathsf{PMC}_{\mathbf{E}}$:

$$\mathsf{R}_2^*: \frac{X \vdash ?\{A_1, \ldots, A_{i-1}, A_i, A_{i+1}, \ldots, A_n\}}{X \vdash ?\{A_1, \ldots, A_{i-1}, C, A_{i+1}, \ldots, A_n\}} \qquad \text{where } (A_i \leftrightarrow C) \in \mathsf{CPL}, 1 \leq i \leq n,$$

provided that $C \neq A_j$ for $j = 1, \ldots, i-1$ and $j = i+1, \ldots, n$.

$$\mathsf{R}_1^*: \frac{X \vdash ?\{A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n\} \qquad X \vdash ?\{A_1, \ldots, A_{i-1}, C, A_{i+1}, \ldots, A_n\}}{X \vdash ?\{A_1, \ldots, A_{i-1}, B \wedge C, A_{i+1}, \ldots, A_n\}}$$

provided that $(B \wedge C) \notin \{A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n\}$.

Due to the presence of rule R_4, the following is a derived rule as well:

$$\mathsf{R}_4^*: \frac{X \vdash ?\{A_1, \ldots, A_n\}}{X \vdash ?\{A_1, \ldots, A_{i-1}, A_j, A_{i+1}, \ldots, A_{j-1}, A_i, A_{j+1}, \ldots, A_n\}}$$

5.2. Some Admissible Rules

As Theorem 14 shows, the system $\mathsf{PMC}_{\mathbf{E}}$ is complete w.r.t. question evocation by finite sets of wffs. Thus, one can make use of some facts concerning question evocation in designing admissible rules of the system.

For conciseness, we abbreviate:

$$?\{A_1, \ldots, A_n\} \tag{27}$$

as

$$?[A_{|n}] \tag{28}$$

and

$$?\{A_1, \ldots, A_n, B_1, \ldots, B_j\} \tag{29}$$

as

$$?[A_{|n}, B_1, \ldots, B_j] \tag{30}$$

The following:

$$?[A_{|n}, B] \tag{31}$$

abbreviates a question of the form:

$$?\{A_1, \ldots, A_n, B\} \tag{32}$$

As for (32), it is assumed that $n > 1$, and similarly in the remaining cases, also below. An expression of the form:

$$?[A_{|n} \otimes B] \tag{33}$$

where $\otimes$ is a binary connective, abbreviates:

$$?\{A_1 \otimes B, \ldots, A_n \otimes B\} \tag{34}$$

and analogously for:

$$?[B \otimes A_{|n}] \tag{35}$$

Let us present some examples of admissible rules of $\mathsf{PMC}_{\mathbf{E}}$.

We have:

Fact 15. Let $n > 1$. If $\mathbf{E}(X, ?[A_{|n}, B])$ and $\mathbf{E}(X, ?[A_{|n}, \neg B])$, then $\mathbf{E}(X, ?[A_{|n}])$.

The corresponding admissible rule is:

$$\mathsf{R}_{cut_r}\colon \frac{X \vdash ?[A_{|n}, B] \qquad X \vdash ?[A_{|n}, \neg B]}{X \vdash ?[A_{|n}]}$$

Fact 16. If $\mathbf{E}(X \cup \{B\}, ?[A_{|n}])$ and $\mathbf{E}(X \cup \{\neg B\}, ?[A_{|n}])$, then $\mathbf{E}(X \vdash ?[A_{|n}])$.

Thus, we get:

$$\mathsf{R}_{cut_l}\colon \frac{X, B \vdash ?[A_{|n}] \qquad X, \neg B \vdash ?[A_{|n}]}{X \vdash ?[A_{|n}]}$$

Fact 17. If $\mathbf{E}(X \cup \{B\}, ?[A_{|n}])$ and $(B \leftrightarrow C) \in \mathsf{CPL}$, then $\mathbf{E}(X \cup \{C\}, ?[A_{|n}])$.

Hence, the following rule is admissible:

R_{leqv}: $\dfrac{X, B \vdash ?[A_{|n}]}{X, C \vdash ?[A_{|n}]}$ where $(B \leftrightarrow C) \in \mathsf{CPL}$.

Fact 18. If $\mathbf{E}(X \cup \{B\}, ?[A_{|n}])$, then $\mathbf{E}(X, ?[B \to A_{|n}])$.

Therefore, we have an admissible rule which is, in a sense, a "converse" of rule R_3:

R_{3r}: $\dfrac{X, B \vdash ?[A_{|n}]}{X \vdash ?[B \to A_{|n}]}$

Fact 19. If $\mathbf{E}(X, ?[A_{|n} \to B])$, then $\mathbf{E}(X \cup \{\neg B\}, ?[\neg A_{|n}])$.

Therefore, we get:

$\mathsf{R}_{\neg \to_r}$: $\dfrac{X \vdash ?[A_{|n} \to B]}{X, \neg B, \vdash ?[\neg A_{|n}]}$

Following Smullyan [21], we introduce the concepts of α- and β-wffs (*cf.* Table 1). However, we do not consider double negated formulas as α-wffs.

Table 1. α / β wffs.

α	α_1	α_2	β	β_1	β_2
$A \wedge B$	A	B	$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	A	B
$\neg(A \to B)$	A	$\neg B$	$A \to B$	$\neg A$	B

α- and β-wffs are CPL-wffs. Table 1 assigns to an α-wff two CPL-wffs, α_1 and α_2, such that the α-wff is true (under a CPL-valuation) if α_1 *and* α_2 are true (under the valuation). Moreover, Table 1 assigns to a β-wff two wffs, β_1 and β_2, such that the β-wff is true (under a CPL-valuation) if β_1 *or* β_2 is true (under the valuation).

One can easily extract the corresponding admissible rules from Table 1 and the following:

Fact 20. If $\mathbf{E}(X \cup \{\beta_1\}, ?[A_{|n}])$ and $\mathbf{E}(X \cup \{\beta_2\}, ?[A_{|n}])$, then $\mathbf{E}(X \cup \{\beta\}, ?[A_{|n}])$.

Fact 21. If $\mathbf{E}(X \cup \{\alpha_1, \alpha_2\}, ?[A_{|n}])$, then $\mathbf{E}(X \cup \{\alpha\}, ?[A_{|n}])$.

Fact 22. If $\mathbf{E}(X \cup \{\alpha\}, ?[A_{|n}])$, then $\mathbf{E}(X \cup \{\alpha_1, \alpha_2\}, ?[A_{|n}])$.

Fact 23.
If $\mathbf{E}(X, ?[A_{|n}, \beta])$, $\beta_1 \neq \beta_2$, and $A_i \neq \beta_j$, where $j = 1, 2$ and $i = 1, \ldots, n$, then $\mathbf{E}(X, ?[A_{|n}, \beta_1, \beta_2])$.

Fact 24.
If $\mathbf{E}(X, ?[A_{|n}, \alpha_1])$ as well as $\mathbf{E}(X, ?[A_{|n}, \alpha_2])$, and $A_i \neq \alpha$ for $i = 1, \ldots, n$, then $\mathbf{E}(X, ?[A_{|n}, \alpha])$.

Fact 25. If $\mathbf{E}(X \cup \{B\}, ?[A_{|n}])$ and $\neg B \neq A_i$ for $i = 1, \ldots, n$, then $\mathbf{E}(X, ?[A_{|n}, \neg B])$.

Let us also note that we have $\mathbf{E}(X, ?\{A, \beta_1, \beta_2\})$ if $\mathbf{E}(X, ?\{A, \beta\})$ holds, and β_1, β_2, A are pairwise syntactically distinct. Moreover, we have $\mathbf{E}(X, ?\{A, \alpha\})$ if $\mathbf{E}(X, ?\{A, \alpha_1\})$ and $\mathbf{E}(X, ?\{A, \alpha_2\})$ hold, and $A \neq \alpha$.

6. Conclusions

IEL considers not only inferences with declarative premises and interrogative conclusions, but also inferences with interrogative premises and conclusions. Validity of such inferences is defined in IEL in terms of *erotetic implication*, being a ternary relation between a question, a (possibly empty) set of declarative formulas, and a question. However, no axiomatic system whose theorems describe erotetic implication is known so far. Developing such system(s) constitutes an interesting challenge.

We have remained here at the propositional level. One can argue that most (if not all) of propositional questions analyzed in the literature can be paraphrased by expressions falling under the schema (6) and thus formalized, at the CPL-level, by expressions of the form (5). This sheds some light on the importance of the completeness result for $\mathsf{PMC}_{\mathbf{E}}$ (*i.e.*, Theorem 14). A natural step further would be to consider the first-order case. This would require an incorporation of the so-called constituent questions (*which-*, *what-*, *where-*, *when*-questions, and so forth). However, providing a formal representation of some of them, in particular multiple wh-questions (e.g., "Who knows where Mary bought what?"), is not an easy task. Constituent questions are formalized differently in different theories. Moreover, it is doubtful if there exists an unique paraphrase pattern for all these questions. This suggests that in the first-order case one might hope only for a variety of axiomatic systems for question evocation or erotetic implication, with completeness results (if any) of a value restricted to the class of constituent questions just considered.

The last remark is this. Although IEL gave rise to the so-called method of Socratic proofs and some logical calculi (for Classical Logic as well as non-classical logics) which are useful in proof-search (see e.g., [22–25]), the system $\mathsf{PMC}_{\mathbf{E}}$, pertaining to one of the central notions of IEL, is barely useful in this respect. This is due to the presence of rule R_2. It is still an open problem how one can develop systems for question evocation that facilitate proof-search.

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