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A NOTE ON DIAGONALIZATION

Abstract

We present a diagonal method of constructing a denumerable family of infinite recursive subsets of a recursive set all of which are different from elements of an effectively given family of infinite r.e. subsets of this set. The construction in question leads to some incompleteness results, e.g. in problem solving systems.

0. Let us consider a structure of the form (Φ, Ψ, f), where Φ and Ψ are infinite subsets of N (= the set of natural numbers), Φ is a recursive set, Ψ is an r.e. (recursively enumerable) set, and $f : \Psi \times N \to \Phi$ is a partial recursive function. Let:

$$rng_n(f) = \{y : y = f(n, x) \text{ for some } x \in N\}.$$

Observe that $rng_n(f)$ equals the range of a function f_n defined as follows:

$$f_n(x) = f(n, x),$$

where $n \in \Psi$. Since f is a partial recursive function, each f_n is a partial recursive function. The range of f as well as the range of any function f_n defined above are r.e. subsets of Φ.

We impose the following condition on the function f:

(†) $rng_n(f)$ is a (countably) infinite set, for all $n \in \Psi$.

Condition (†) is equivalent with the following:

(‡) for each $n \in \Psi$, the range of f_n is a (countably) infinite set.

Our aim can now be described as follows. We assume that the condition (†) holds. Then we present a method of constructing an infinite family of total strictly increasing recursive functions whose ranges are different from the ranges of any of the recursive functions f_n already given, but are still subsets (as a matter of fact, infinite recursive subsets!) of the initial set Φ. In this way we construct an infinite family of ,,new" infinite recursive subsets of Φ, that is, infinite recursive subsets of Φ which are different from any set of the form $rng_n(f)$, for all $n \in \Psi$.

We dare to think that the non-standard diagonalization method we apply in our proof is worth some attention.

The range of a function φ will be designated by $rng(\varphi)$. By $\varphi[Y]$ we designate the image of a set Y under a function φ.

1. First, we shall consider the most demanding case: both Φ and Ψ are proper subsets of N. We reason as follows.

By assumption, Ψ is an infinite r.e. subset of N. Hence there exists a recursive bijection $k : N \to \Psi$. Moreover, the domain of f is $\Psi \times N$. Thus the function g defined by:

$$g(m, x) = f(k(m), x)$$

is a total recursive function.

For each $m \in N$, we define:

$$g_m(x) = g(m, x).$$

By assumption, Φ is an infinite recursive set. So there exists a recursive bijection $h : N \to \Phi$ such that h is a total strictly increasing recursive function. Now we define a partial recursive function $h^* : \Phi \to N$, the converse of h, by the conditions:

$h^*(x) =$ the unique y such that $h(y) = x$, if $x \in \Phi$,
$h^*(x)$ is undefined otherwise.

Note that h^* is strictly increasing.

Observe that $h^*(g(m, x))$ is defined for any $m, x \in N$, since $rng(g) \subseteq \Phi$, and g is total. We define a recursive predicate R by putting:

$$\forall y \forall x \forall m \ (R(y, m, x) \equiv (y < h^*(g(m, x)))).$$

Due to the condition (†) assumed above, for any $y, m \in N$ there exists $x \in N$ such that $R(y, m, x)$. Since, by (†), the set of values of f_n is infinite for any $n \in \Psi$, so is the set of values of g_m for any $m \in N$ as well as its image by h^*. Thus the minimum used in the following definition of the function r:

$$r(y, m) = \mu x \ (R(y, m, x))$$

is effective and therefore r is a total recursive function.

At the consecutive step we choose an arbitrary but fixed natural number $j > 0$ and we define the function t_j as follows:

$t_j(0) = h^*(g(0, 0)) + j$
$t_j(i + 1) = h^*(g(i + 1, r(t_j(i), i + 1))) + j.$

Note that the second clause amounts to:

$$t_j(i + 1) = h^*(g(i + 1, \mu x(t_j(i) < h^*(g(i + 1, x))))) + j.$$

Thus, generally speaking, in order to calculate $t_j(i + 1)$ we first calculate the smallest element, say, a, of $rng(g_{i+1})$ such that $h^*(g_{i+1}(a))$ is greater than $t_j(i)$, and second, we add j to the value of $h^*(g_{i+1}(a))$. Observe that the number $h^*(g_{i+1}(a))$ lies between $t_j(i)$ and $t_j(i + 1)$. This constitutes the core of our diagonal construction. The existence of the appropriate number a is warranted by the assumption (†) and by the definition of g.

A value of t_j need not belong to Φ. As we will see, however, this does not make any harm.

Since t_j is defined in terms of recursive functions and operations which lead from recursive functions to recursive functions, t_j is a recursive function. Note that t_j is a total recursive function (recall that $h^*(g(m, x))$ is defined for any $m, x \in N$). Observe also that t_j is a strictly increasing recursive function, for the construction gives:

(♡) $t_j(x) > t_j(y)$ if $x > y$.

Thus the range of t_j, $rng(t_j)$, is an infinite recursive set.

We will show that $rng(t_j) \neq h^*[rng(g_m)]$ for any $m \in N$. Let $m = 0$. Now $t_j(0) = h^*(g(0, 0)) + j$. Clearly $g(0, 0) \in rng(g_0)$. Therefore $h^*(g(0, 0)) \in h^*[rng(g_0)]$. On the other hand, $h^*(g(0, 0)) < t_j(0)$, and thus, by (♡), $h^*(g(0, 0)) < y$ for any $y \in rng(t_j)$. Therefore $h^*(g(0, 0)) \notin rng(t_j)$ and hence $rng(t_j) \neq h^*[rng(g_0)]$.

Let $m > 0$. Thus there exists $i \geqslant 0$ such that $m = i + 1$. Recall that $t_j(i+1) = b + j$, where:

$$b = h^*(g(i+1, r(t_j(i), i+1))) = h^*(g(i+1, \mu x\ (t_j(i) < h^*(g(i+1, x))))).$$

Clearly $b \in h^*[rng(g_{i+1})]$. It is easily seen, however, that b is not $t_j(i+1)$. Moreover, $b > t_j(i)$, and thus, by (♡), b is not $t_j(l)$ for $l = 1, \ldots, i$. On the other hand, $b < t_j(i+1)$ and hence, by (♡) again, b is not $t_j(e)$ for any $e > i + 1$. Therefore $b \notin rng(t_j)$. Hence $rng(t_j) \neq h^*[rng(g_{i+1})]$.

Thus for all $m \in N$, $rng(t_j) \neq h^*[rng(g_m)]$.

Finally, let us define the following function $w_j : N \to \Phi$

$$w_j(x) = h(t_j(x)).$$

Since w_j is the superposition of h and t_j, and these are recursive functions, w_j is a recursive function as well. Since both h and t_j are strictly increasing recursive functions, w_j is a strictly increasing recursive function. It is easily seen that w_j is total. Moreover, since h is a bijection from N to Φ, the range of w_j, i.e. $rng(w_j)$, is a subset of Φ. Thus, by the properties of w_j, the range of w_j is an infinite recursive subset of Φ.

Now we will show that $rng(w_j) \neq rng(g_m)$ for any $m \in N$.

We have already proved that $rng(t_j) \neq h^*[rng(g_m)]$. But $rng(w_j) = h[rng(t_j)]$ and $h[h^*[rng(g_m)]] = rng(g_m)$, for h^* is the converse of h. Therefore $rng(w_j) \neq rng(g_m)$ for any $m \in N$, as required.

Recall that g is defined by: $g(m, x) = f(k(m), x)$, where k is a recursive bijection from N to Ψ. Thus we have $\{X : X = rng(g_m)$ for some $m \in N\} = \{X : X = rng(f_n)$ for some $n \in \Psi\}$.

Therefore we get:

(♣) $rng(w_j) \neq rng(f_n)$, for any $n \in \Psi$.

Since the following holds: $\{X \subseteq \Phi : X = rng(f_n)$ for some $n \in \Psi\} = \{X \subseteq \Phi : X = rng_n(f)$ for some $n \in \Psi\}$, we also have

(♠) $rng(w_j) \neq rng_n(f)$, for any $n \in \Psi$.

Now recall that j has been an arbitrary natural number greater than 0. So we can say that the family of functions $(w_j)_{j \in N - \{0\}}$ is denumerable.

On the other hand, it is clear that the exact value of j plays no role in showing that the condition (♠) holds. Therefore the family $(w_j)_{j\in N-\{0\}}$ is an infinite family of total strictly increasing recursive functions whose ranges are different from the range of any of the recursive function f_n, but are still subsets of the initial recursive set Φ.

Finally, consider $w, w' \in (w_j)_{j\in N-\{0\}}$ such that $w \neq w'$. There are two possibilities: either $w(0) < w'(0)$ or $w'(0) < w(0)$. Since both w and w' are strictly increasing, it follows that in each case the sets of values of w and w' are different. In other words, for any $w, w' \in (w_j)_{j\in N-\{0\}}$ we have $rng(w) \neq rng(w')$ if $w \neq w'$.

Thus we have constructed an infinite family of recursive subsets of Φ which are different from the ranges of the already given recursive functions f_n.

2. So far we have considered the most demanding case. Observe, however, that when $\Phi = N$, the required construction simplifies a lot, since we need neither h nor h^*. Now w_j can be defined immediately by the conditions:

$$w_j(0) = g(0,0) + j$$
$$w_j(i+1) = g(i+1, \mu x\ (w_j(i) < (g(i+1,x)))) + j.$$

When $\Psi = N$, we do not need g (that is, g coincides with f in such a case).

3. Anyway, we have just proved the following:

Theorem.

Let $\mathfrak{A} = (\Phi, \Psi, f)$ be such that: Φ is an infinite recursive set, Ψ is an infinite r.e. set, and $f : \Psi \times N \to \Phi$ is a partial recursive function. Let:

$$rng_n(f) = \{y : y = f(n,x) \text{ for some } x \in N\}$$

$$\Sigma_{\mathfrak{A}} = \{X \subseteq \Phi : X = rng_n(f) \text{ for some } n \in \Psi\}.$$

If each element of the family $\Sigma_{\mathfrak{A}}$ is an infinite set, then there exists an infinite family of infinite recursive subsets of Φ which are different from any of the sets in $\Sigma_{\mathfrak{A}}$.

For any $X \subseteq N$ let Δ_X be the family of all infinite recursive subsets of X. We say that $\mathfrak{A} = (\Phi, \Psi, f)$ is *recursively complete*, if $\Delta_\Phi \subseteq \Sigma_{\mathfrak{A}}$.

The theorem proved above shows that no structure $\mathfrak{A} = (\Phi, \Psi, f)$ satisfying the assumptions of the theorem is recursively complete.

4. A few words comparing the above procedure with commonly known results are in order. Most frequently, incompleteness results are obtained with the help of some form of the Diagonal Lemma (cf. e.g. Hinman's monograph [2], 320). For any $R \subseteq A \times A$ and any $a \in A$ let:

$$R_a = \{b \in A : R(a, b)\}.$$

Define the R-diagonal of A to be:

$$D_R = \{a \in A : R(a, a)\}.$$

Then the Diagonal Lemma says that for any $R \subseteq A \times A$ the set $A - D_R$ is not equal to any R_a, for $a \in A$.

Call the sets R_a the R-sections, a being an R-index for R_a. Call R universal for the family $\mathbb{C} \subseteq \wp(A)$, if each element of $\mathbb{C}$ is an R-section. Then we have e.g. the following well known applications of the Diagonal Lemma:

- For any set A there is no surjective mapping $F : A \to \wp(A)$. Let $R^F = \{(a, b) : b \in F(a)\}$. Then the range of F equals exactly the family of all R^F-sections. By the Diagonal Lemma, the complement of the R^F-diagonal set is not an R^F-section and hence F is not surjective.
- For any theory T in the language of Peano Arithmetic PA let $\{\varphi_a : a \in N\}$ be the list of all formulas of T and let $\overline{m}$ be the numeral corresponding to the natural number m. Then the relation $R^T = \{(a, m) : \varphi_a(\overline{m}) \in T\}$ is universal for the family of all weakly T-representable sets (observe that each R^T-section R^T_a is a set weakly T-representable by φ_a). As a consequence of the Diagonal Lemma we get that the complement of the diagonal set $\{m : \varphi_m(\overline{m}) \in T\}$, i.e. the set $\{m : \neg\varphi_m(\overline{m}) \in T\}$ is not weakly T-representable. This leads to the conclusion that if T is a consistent theory including PA, then T is not decidable.
- Similarly, if we put $R^{Th(PA)} = \{(a, m) : \varphi_a(\overline{m}) \in Th(PA)\}$, where $Th(PA)$ is the set of all sentences that are true in the standard model of PA, we come to the conclusion that $R^{Th(PA)}$ is universal for the class of all sets definable over PA which in turn, by Diagonal Lemma,

implies the undefinability of arithmetical truth: the set $Th(PA)$ is not definable in PA.

The construction presented above in **1.–3.** is not explicitly of this form. Rather, it is akin to those approaches in which one makes essential use of some infinity assumptions. For example, consider a result reported by Cori and Lascar [1], Part II, 59–60. Let F be a two-argument recursive function and assume that for all $x \in N$ the set $A_x = \{F(x, y) : y \in N\}$ is infinite. Then there exists an infinite recursive set B distinct from all the sets A_x, for $x \in N$. In order to prove this, we define the function G:

$G(0) = 0$
$G(x + 1) = (\mu z\ ((z)_2 = F(x, (z)_1) \wedge G(x) < (z)_2))_2 + 1.$

Here $(z)_1$ and $(z)_2$ are, correspondingly, the first and the second projection defined in the usual manner for the Cantor's pairing function:

$$\langle x, y\rangle = \frac{(x + y)^2 + 3x + y}{2}.$$

The function G is a strictly increasing total recursive function. Let B be its range. Hence B is an infinite recursive set. Now, B is distinct from all A_x, because:

- we have $G(x) < (\mu z\ ((z)_2 = F(x, (z)_1) \wedge G(x) < (z)_2))_2 < G(x + 1)$ and hence $(\mu z\ ((z)_2 = F(x, (z)_1) \wedge G(x) < (z)_2))_2$ does not belong to B, i.e. the range of G;
- $(\mu z\ ((z)_2 = F(x, (z)_1) \wedge G(x) < (z)_2))_2 \in A_x$ for all $x \in N$.

This construction is a special case of the construction presented in this paper (cf. **2** above).

5. Let us shortly comment about certain possible applications of the construction just presented. First of all, note that the following are immediate consequences of the theorem.

1. Let $\mathfrak{A} = (\Phi, \Psi, f)$ be such that Φ is an infinite recursive set, Ψ is an infinite r.e. set, $f : \Psi \times N \to \Phi$, each element of the family $\Sigma_{\mathfrak{A}}$ is an infinite set, and $\Delta_\Phi \subseteq \Sigma_{\mathfrak{A}}$. Then the function f is not recursive.
2. Let $\mathfrak{A} = (\Phi, \Psi, f)$ be such that Φ is an infinite set of natural numbers, Ψ is an infinite r.e. set, $f : \Psi \times N \to \Phi$ is a partial recursive function and $\Delta_\Phi \subseteq \Sigma_{\mathfrak{A}}$. Then the set Φ is not recursive.

The above consequences, as well as the theorem itself serve as examples of incompleteness results, it the wide sense of the word. Let us mention one — out of many possible — interpretation relevant in this respect.

Take a structure of the form (Φ, Ψ, f). Suppose that Φ represents a set of (suitable numerical codes of) *solutions* of a problem P. Suppose further that Ψ is a denumerable r.e. set of (numerical codes of) *conditions*, N is the set of indices of possible worlds, and $f : \Psi \times N \to \Phi$ is a partial recursive function which, intuitively speaking, assigns to a condition and a world the solution of P which is true in the world and satisfies the condition. We may say that the set:

$$\Gamma_c = \{y \in \Phi : y = f(c, x) \text{ for some } x \in N\}$$

is the set of solutions of P that are *determined by* condition $c \in \Psi$.

Assume that Φ is a denumerable set, and that Γ_c is a countably infinite set, for all $c \in \Psi$. Now we face the following dilemma:

- Φ is a recursive set. Hence, by the theorem, there exist infinite recursive subsets of Φ (actually, infinitely many of them!) such that these sets are not determined by any condition specified in the r.e. set Ψ. In other words: *there exist infinite decidable sets of solutions of P which are subsets of the already given decidable set of solutions, but are not determined by any of the effectively listed conditions.* Note that we neither assume nor deny that Φ represents the set of all solutions of P; note also that the above effect is persistent in the sense that no effective enrichment of Ψ changes the picture.
- Φ has infinite recursive subset(s) and each infinite recursive subset of Φ is determined by some condition which belongs to Ψ. Then, according to the theorem, Φ is not a recursive set. In other words: *each infinite decidable set of solutions of P included in an already given set of solutions S of P is determined by some of the effectively listed conditions, but the whole set of solutions S is not decidable.* As before, we neither assume nor deny that S is the set of all solutions of P. However, assuming that this is the case makes the effect really intriguing.

Some further interpretations of the structures of the form $\mathfrak{A} = (\Phi, \Psi, f)$ satisfying the effective conditions described at the beginning of the paper may be obtained in terms of provability relation or in terms of Turing machines.

References

[1] R. Cori and D. Lascar, **Mathematical Logic. A Course with Exercises**, Oxford University Press, Oxford, 2001.
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