An Essay on Inferential Erotetic Logic

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1 Introduction

By and large, Inferential Erotetic Logic (IEL for short) is an approach to the logic of questions which puts in the centre of attention inferential aspects of questioning. IEL is not an enterprise of the last few years only. The idea originates from the late 1980s. It evolved through time. Initially, the stress was put on the phenomenon of question raising. This changed gradually, as some forms of reasoning that involve questions have appeared to be analysable by means of the conceptual apparatus developed.[1] In this essay I present the basics of IEL and comment on them. Most, though not all, of the ideas discussed here have been scattered across my earlier publications. The invitation from the organizers of the *Asking and Answering* workshop (Greifswald, September 2020) resulted in an attempt of presenting the themes of IEL in a concise but, as I hope, also comprehensible way.

2 Erotetic Inferences

As Sylvain Bromberger puts it:

> *We ask questions for all sorts of reasons and with many different purposes in mind – e.g., to test someone's knowledge, to offer someone the opportunity to show his erudition, to kill time, to attract attention; but questions have one basic function, the asking for information not already in our possession.* [2], p. 86.

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[1]The monograph [38] summarizes results obtained until the early 1990s. It is concerned, mainly, with first-order languages enriched with questions and supplemented with the standard model-theoretic semantics. The book [44] presents IEL in a more general conceptual setting and overviews new results received till the date of publication.

Yet, before a question is asked or posed, one has to arrive at it. In many cases arriving at questions resembles coming to conclusions: there are premises involved and some inferential thought processes take place. In other words, there exist *erotetic inferences*, that is, thought processes in which one arrives at a question on the basis of some previously accepted declarative sentence(s) and/or a previously posed question. Consider:

(I) *There is a cat in this room.*
Someone let it in.

Who let the cat in?

and

(II) *Where did Andrew leave for?*
If Andrew took his famous umbrella, then he left for London.
If Andrew did not take his famous umbrella, then he left for Paris or Moscow.

Did Andrew take his famous umbrella?

As for (I), the set of premises contains declarative sentences only, while in the case of (II) the set of premises comprises declarative sentences and a question. It also happens that no declarative premise occurs, *viz*:

(III) *Did Andrew fly by BA, or by Ryanair, or by neither?*

Did Andrew fly by BA?

It can be shown that erotetic inferences are subjected to patterns. This makes their logical analysis possible.

Remark 1. One should differentiate between inferences *about* questions and inferences *with* questions. What I have called above erotetic inferences belongs to the latter category. Here are examples of inferences about questions:

> *One cannot divide by zero. Therefore the question "What is the value of* $4 \div 0$*?" makes no sense.*
>
> *The question "How did you solve this problem?" has no answer since you did not solve it. Therefore the question "When did you solve the problem?" is pointless.*

IEL concentrates on inferences with questions.

2.1 The Validity Issue

Some erotetic inferences are *intuitively valid*, while others are not. The following can serve as a preliminary test of intuitive validity: put the expression "so the question arises:" just before the conclusion. If the outcoming description of an erotetic inference is undoubtedly true, the inference can be regarded as intuitively valid. Obviously, (I), (II), and (III) specified above pass the test. Nevertheless, there are cases in which one does not get indisputable results. The intuitive concept of validity is fuzzy or even vague.

IEL offers an account of *validity* of erotetic inferences. Yet, validity is a normative notion and the issue of naturalistic fallacy cannot be ignored. So in order to define validity for erotetic inferences some – not entirely, but still – arbitrary decisions have to be, and actually are, made. This is not a peculiarity of IEL. Logics of other types of inferences had done the same in their accounts of validity of inferences considered, although we usually disregard it as we got used to solutions established in the past.

IEL proceeds as follows. First, some criteria of validity are proposed, separately for erotetic inferences that involve only declarative premises and for these in which an interrogative premise occurs. Once criteria of validity are set, two semantic relations are defined: *evocation* of questions by sets of declarative sentences/formulas, and *erotetic implication* of a question by a question together with a set of declarative sentences/formulas. Validity of erotetic inferences of the consecutive kinds is defined in terms of question evocation and erotetic implication, respectively. Although these concepts differ[2], there is a unifying idea behind defining validity by means of them: each concept is a formal counterpart of the corresponding notion of question raising. The definition of question evocation provides an explication of the intuitive notion "a question arises from a set of declarative sentences." The definition of erotetic implication, in turn, is an explication of the intuitive notion "a question arises from a question on the basis of a set of declarative sentences."[3] It is neither assumed nor denied that the relevant set of declaratives is non-empty. The case in which it is empty allows us to deal with validity of erotetic inferences which do not involve any declarative premises (cf., e.g., (III) above). Alternatively, one can use here the concept of *pure erotetic implication*, being a binary relation between questions.

[2]However, surprisingly enough, it is possible to define question evocation, in a somewhat tricky way, in terms of erotetic implication; cf. [44], p. 84.

[3]Let me stress: IEL does not provide analytic definitions of the above concepts of "arising", but their *explications*; cf. [38], Chapter 1.

3 Syntax and Semantics: General Insights

3.1 Syntax

IEL employs formal languages, in which at least two categories of well-formed expressions occur: *declarative well-formed formulas* (d-wffs for short) and *erotetic well-formed formulas* (hereafter: e-wffs or simply *questions*). Generally speaking, an object-level formal language employed has thus (possibly among others) a "declarative part" and an "erotetic part." The former can be a (modal or non-modal) propositional language, a first-order language (augmented with modalities or not), a higher-order language, etc.

As for questions, IEL prefers a semi-reductionistic approach. The general idea is: e-wffs fall under the following schema:

$$?\Theta$$

where Θ is an expression of the object-level formal language such that Θ is equiform with the expression of the metalanguage which, in turn, designates the set of *direct answers* to the e-wff.

E-wffs (to be more precise, some of them) are formal counterparts of natural-language questions. An e-wff Q *represents* a natural-language question Q^* *construed in such a way* that direct (i.e. immediate and sufficient) answers to Q^* are represented by the direct answers to Q.[4]

Here is an example how the semi-reductionistic approach works. When we add the question mark ? and the brackets: {, } to the vocabulary of a formal language, we can enrich the language with e-wffs of the form:

$$?\{A_1,\ \ldots, A_n\} \tag{1}$$

where $n > 1$ and $A_1,\ \ldots, A_n$ are pairwise syntactically distinct d-wffs of the initial language; these d-wffs are supposed to be the only direct answers to the e-wff/question.

This is only an example. It cannot be said that each e-wff considered in IEL falls under the schema (1). The semi-reductionistic approach copes with questions having infinitely many direct answers by defining, at the metalanguage level, different infinite sets of d-wffs of required kinds, and then by introducing into an object-level language expressions equiform with the respective metalanguage expressions just defined. For examples, see [38], Chapter 3. However, IEL is not committed to the semi-reductionistic approach to questions sketched above. One can work within IEL and introduce e-wffs by applying other patterns known from the literature (for instance, following Kubiński's approach [11] or Belnap's proposals [1]). What is needed

[4] A reader not familiar with logics of questions should bear in mind that, at both levels, being true *is not* a prerequisite for being a direct answer.

is a formal language such that: (i) d-wffs and e-wffs occur among its well-formed expressions, where e-wffs are distinct from well-formed expressions of other categories, and (ii) direct answers are assigned, in some way or another, to e-wffs. We are free in designing such a language, but not completely free. Some global constraints are supposed to be met. Here are examples:

- each e-wff has at least two direct answers;
- direct answers are sentences, i.e. d-wffs with no individual or higher-order free variables.[5]

One has to bear in mind that the claims of IEL rely upon, among others, stipulations of the above kind. In practice, this means that when definitions and theorems of IEL refer to questions, the tacit assumption is that only questions which meet the respective stipulations are referred to.

3.2 Semantics

IEL does not assume that questions/e-wffs are true or false. Semantic properties of and relations between e-wffs are defined in a way that takes as a prerequisite the existence of assignments of (sets of) direct answers to e-wffs In order to proceed at the general level, only a semantics of the declarative part of a language is needed. It should be rich enough to define concepts of truth and entailment for d-wffs. A detailed semantic account of e-wffs/questions themselves brings an added value, but definitions of basic notions of IEL do not rely upon any elaborated semantics of questions. Similarly, in its general setting IEL remains neutral in the controversy as to what "The Logic" of declaratives is. One can use either Classical Logic or a non-classical logic. But different logics have diverse semantics. A unifying framework is provided by *Minimal Erotetic Semantics* (MiES), within which entailment relations determined by different logics can be simulated. Yet, for space reasons, I will not present MiES here.[6]

So let us only assume that the declarative part of a formal language considered is supplied with a semantics rich enough to define some relativized (to a semantic item, such as a valuation, a model of an appropriate kind, and so forth, depending on the logic and its semantics chosen) concept of truth for d-wffs. Having the concept of truth, one can define entailment. It is convenient to operate with the concept of multiple-conclusion entailment (mc-entailment for short), being a relation between *sets* of d-wffs (cf. [31]). The idea is: a set of declarative sentences, X, mc-entails a set of declarative sentences, Y, iff the hypothetical truth of all the sentences in X warrants the existence of a true sentence in Y.[7] For instance, the set:

[5] As for propositional languages, direct answers are propositional formulas.

[6] For MiES, an interested reader can consult, e.g., [44], chapters 3 and 4.

[7] The expression "iff" abbreviates, here and below, 'if and only if.'

> $\{$*There is a cat in this room. Either Andrew, or Paul, or Dorothy let the cat in.*$\}$

mc-entails the set:

> $\{$*Andrew let the cat in. Paul let the cat in. Dorothy let the cat in.*$\}$

We use the symbol $\Vdash$ for mc-entailment. As for formal languages considered, definition of $\Vdash$ falls under the following schema:

> (Mc-entailment) $X \Vdash Y$ iff for each $\mathcal{M} \in (\ldots)$: if all the d-wffs in X are true in $\mathcal{M}$, then at least one d-wff in Y is true in $\mathcal{M}$.

where $\mathcal{M}$ refers to a semantic item in relation to which truth of d-wffs is defined, and the ellipsis should be filled with an expression denoting a class of such items. Needless to say, single-conclusion entailment, $\models$, can be defined by: $X \models A$ iff $X \Vdash \{A\}$.

Here are examples of semantic concepts pertaining to e-wffs/questions, defined within the framework sketched above:

(Soundness of a question) An e-wff Q is *sound* in $\mathcal{M}$ iff at least one direct answer to Q is true in $\mathcal{M}$.

(Presupposition) A d-wff A is a *presupposition* of an e-wff Q iff A is entailed by each direct answer to Q.

Note that a question having false presupposition(s) has no true direct answer. IEL neither ignores the existence of loaded questions nor stipulates that negations of presuppositions always count as direct answers.

Here are some further useful notions:

(Prospective presupposition) A d-wff A is a *prospective presupposition* of an e-wff Q iff A is a presupposition of Q and the set of direct answers to Q is mc-entailed by the (singleton set comprising) the presupposition A.

(Normal question) An e-wff Q is *normal* iff Q must have a true direct answer if its presuppositions are all true.[8]

4 Question Evocation and Erotetic Implication

We are now ready to introduce the concepts of question evocation and erotetic implication. In order to facilitate reading, the proposed definitions will be illustrated with natural-language examples, and short comments, expressed in general terms, will be added.

By $\mathbf{d}Q$ we designate the set of direct answers to a question/e-wff Q. The symbol $\Vdash$ stands for mc-entailment (cf. subsection 3.2).

[8] More formally: Q has presuppositions, and the set of direct answers to Q is mc-entailed by the set of presuppositions of Q.

4.1 Question Evocation

The expression $\mathbf{E}(X, Q)$ abbreviates "a set of d-wffs X evokes a question/e-wff Q."

Definition 1 (Question evocation). $\mathbf{E}(X, Q)$ *iff*

1. $X \Vert\!\models \mathbf{d}Q$*, and*
2. *for each* $A \in \mathbf{d}Q : X \Vert\!\not\models \{A\}$*.*

For instance, one can say[9] that the following set of declarative sentences:

$$\{\textit{There is a cat in this room. Someone let it in.}\} \tag{2}$$

evokes the question:

$$\textit{Who let the cat in?} \tag{3}$$

Call a natural-language question *sound* if at least one direct answer to the question is true. (Observe that question (3) need not be sound. It is construed here as not allowing "No one" and its equivalents as *direct* answers.) Generally speaking, the first clause of Definition 1 amounts to *transmission of truth into soundness*: if only X consists of truths, the question Q must be sound. (Clearly, if only (2) consists of truths – it need not! – there must be someone who let the cat in.) The second clause amounts to the claim that no single direct answer to Q is entailed by X. (Obviously, one cannot decide who let the cat in on the basis of (2) only.)

Remark 2. To put it mildly, mc-entailment is not among concepts well-accustomed by non-logicians (and some logicians, too). Can we avoid referring to mc-entailment when defining question evocation? The answer is negative in the general case, but affirmative in some special cases. For instance, when we operate with a language in which all questions are normal (in the sense specified in subsection 3.2 above), evocation of Q by X can be defined by the following clauses: (i) X entails each presupposition of Q, and (ii) X does not entail any direct answer to Q. For normal questions which have maximal presuppositions (i.e. single presuppositions that entail all the remaining presuppositions), clause (i) can be replaced with (i') X entails a maximal presupposition of Q.

When question evocation is defined according to the pattern provided by Definition 1, clauses (i)–(ii) and clauses (i')–(ii) characterize properties of evocation of normal questions and of normal questions equipped with maximal presuppositions, respectively.

[9] The modality "can" is used here for a reason. Saying this with certainty would require listing assumptions concerning logical representations of the analysed natural-language expressions in a formal language, as well as concerning the underlying logic. For obvious reasons, we skip them here. One should bear in mind that an analogous remark pertains to the remaining natural-language examples presented below.

Remark 3. Definition 1 pertains only to questions whose sets of direct answers (or "principal possible answers" labelled differently) are determined one way or another. Strictly speaking, it pertains to e-wffs of formal languages employed in IEL. It is doubtful if every natural-language question can be analysed to the effect its formal representative is an e-wff having a well-defined set of direct answers. This does not mean, however, that every why-question remains outside the area of applicability of IEL. A reader intrigued by this enigmatic statement is advised to consult [12], [39], and [43].

Remark 4. The second clause of Definition 1 refers to the lack of entailment. This does not lead into troubles when the entailment relation operated with is decidable. However, it need not be so. It happens that entailment is only recursively enumerable, while the lack of entailment is not even recursively enumerable. First-Order Logic entailment (hereafter: FOL-entailment) provides a paradigmatic example here. As a consequence, in such a situation question evocation relation[10] is not recursively enumerable. However, one should not confuse the lack of semidecidability of the whole relation with the impossibility of showing that something is an instance of the relation. An example will be of help. Let P, R be distinct two-place predicates, and a, b be individual constants. Although there is no algorithm which "detects" the lack of FOL-entailment in each case of its occurrence, we can still, by the construction of a countermodel, show that neither $P(a, b)$ nor $R(a, b)$ is FOL-entailed by the disjunction $P(a, b) \vee R(a, b)$. Establishing this, we are able to conclude that the question $?\{P(a, b), R(a, b)\}$ is (FOL-)evoked by the singleton set $\{P(a, b) \vee R(a, b)\}$.

Remark 5. Definition 1 provides an explication of one of intuitive notions of question raising. It does it successfully with respect to the criteria of adequacy of explication previously set (cf. [38], Chapter 1). But questions often arise from inconsistencies. As long as the underlying logic of declaratives validates *Ex Falso Quodlibet* – but please remember that not all logics do! – no question is evoked by an inconsistent set of declaratives. In order to cope with the inconsistency case, one can adopt different strategies (cf., e.g., [26], [27], [46]). There is no room for presenting them here.

Remark 6. One may argue that the concept of question evocation understood according to Definition 1 is too broad. Without discussing this issue, let me only mention that the relations defined below are (interesting) special cases of question evocation defined above.

Definition 2 (Question generation). $\mathbf{G}(X, Q)$ *iff*

1. $X \models\!\!\models \mathbf{d}Q$, *and*

[10] Understood as the set of all ordered pairs $< X, Q >$, where X is a set of wffs of a language, and Q is an e-wff of the language, such that $\mathbf{E}(X, Q)$ holds.

2. for each $A \in \mathbf{d}Q : X \not\Vdash \{A\}$*, and*

3. $\emptyset \not\Vdash \mathbf{d}Q$.

The third clause of Definition 2 supplements the first one: the transmission of truth into soundness effect takes place, but not just due to the fact that the generated question is always sound.

Definition 3 (Strong evocation)**.** $\mathbf{E}^{\star}(X, Q)$ *iff*

1. $X \Vdash \mathbf{d}Q$*, and*

2. for each $A \in \mathbf{d}Q : X \not\Vdash (\mathbf{d}Q \setminus \{A\})$.

The second clause of Definition 3 ensures that no proper subset of the set of direct answers to Q is mc-entailed by X. Hence X strongly evokes Q just in case the hypothetical truth of all the wffs in X warrants that a truth occurs in the whole set of direct answers to Q, but does not warrant this for any proper subset of the set. In the case of languages in which classical disjunction occurs and questions with finite sets of direct answers are the only ones considered, this happens when X entails a disjunction of all the direct answers to Q, yet does not entail any disjunction of some but not all direct answers to the question.

4.1.1 Evocation and Validity

An erotetic inference of the first kind leads from premises being declarative sentence(s) to a conclusion having the form of a question. IEL proposes the following criteria of validity of erotetic inferences of the first kind (these criteria are supposed to be satisfied jointly):

($\mathbf{C}_1$) (Transmission of truth into soundness). *If the premises are all true, then the question which is the conclusion must be sound.*

($\mathbf{C}_2$) (Informativeness). *A question which is the conclusion must be informative relative to the premises.*

There is no room for an extensive presentation of pros and cons of such a solution. An interested reader is advised to consult, e.g., [38], Chapter 8, or [44], Chapter 5.

Taken purely syntactically, an erotetic inference from a set of declaratives X to a question Q is simply the ordered pair $\langle X, Q \rangle$. Assume that both the elements of X and Q are expressions of a formal language for which question evocation has been defined. Given this assumption, we introduce:

Definition 4. *An erotetic inference* $\langle X, Q \rangle$ *is valid iff* $\mathbf{E}(X, Q)$.

Definition 4 pertains only indirectly to erotetic inferences whose premises and conclusions are expressed in a natural language. But this in not unusual. For instance, we often speak about logical entailment between declarative sentences of a natural language, although logical entailment is, strictly speaking, defined for a (corresponding) formal language. Problems with a transition from a natural to a formal language are well-known, and IEL is neither better nor worse in this respect than other formal logics.

Remark 7. Condition $(\mathbf{C}_1)$ of validity and clause 1 of Definition 1 almost mirror each other. Yet, the transition from clause 2 of Definition 1 to condition $(\mathbf{C}_2)$ is not immediate. It relies on the assumption that informativeness of a direct answer w.r.t. a set of declaratives is tantamount to the lack of entailment of the direct answer from the set of declaratives. This works in one direction, but not necessarily in the other: a direct answer entailed by a set of declaratives can be regarded as informative w.r.t. the set when the answer is a "distant consequence" of the set. This gives rise to an issue relevant to the analysis of question raising. However, IEL in its current form simplifies matters in the way presented above. As a consolation, let me only say that, as long as logic is concerned, "is not valid" is not synonymous with "is fallacious."

4.2 Erotetic Implication

Let us now define the second central concept of IEL, namely erotetic implication. The expression $\mathbf{Im}(Q, X, Q_1)$ reads "an e-wff/question Q_1 is erotetically implied by an e-wff/question Q on the basis of a set of d-wffs X."

Definition 5 (Erotetic implication)**.** $\mathbf{Im}(Q, X, Q_1)$ *iff*

1. *for each* $A \in \mathbf{d}Q : X \cup \{A\} \Vdash \mathbf{d}Q_1$*, and*
2. *for each* $B \in \mathbf{d}Q_1$ *there exists a non-empty proper subset* Y *of* $\mathbf{d}Q$ *such that* $X \cup \{B\} \Vdash Y$*.*

For example, the question:

Who let the cat in: Andrew, Dorothy, or Paul? (4)

erotetically implies the question:

Is the cat black, or is it grey? (5)

on the basis of the following set of sentences:

{*Andrew let the cat in iff the cat is black.* (6)
Dorothy or Paul let the cat in iff the cat is grey.}

The first clause of Definition 5 warrants the transmission of soundness and truth into soundness. (There are cats which are neither black nor grey. But if only question (4) is sound and (6) consists of truths, the cat asked about must be either black or grey.) The intuition that underlies the second clause is: each direct answer to an implied question narrows down, together with the respective set X, the class of "possibilities" or "options" offered by the whole set of direct answers to the implying question. (If the cat occurred grey and (6) consists of truths, only two options would remain: Dorothy or Paul. If the cat occurred black and (6) consists of truths, only one possibility would remain, namely Andrew.) Or, to put it differently, each direct answer to an implied question, when added to X, enables us to answer, partially or directly, the implying question.[11]

Let me stress that erotetic implication defined above is, so to say, "Janus-faced." The first clause of its definition "looks forward" (from an implying question to the implied question), while the second clause "looks backward" (from an implied question to the implying question).

The first clause looks suspicious to those who believe that any question is, as a matter of fact, truly answerable. Yet, IEL does not assume anything like this. On the contrary, loaded questions and/or questions carrying factual presuppositions (and thus not necessarily sound) are not ignored.

Remark 8. Speaking about implication usually presupposes a unique "direction of flow." So, maybe, the term "implication" is inaccurate for the semantic relation characterized by Definition 5. However, the term was coined in [36] and is in usage in the field.

Pure erotetic implication, $\mathbf{Im}^{\odot}$, is a binary relation between e-formulas/questions.

Definition 6 (Pure erotetic implication). $\mathbf{Im}^{\odot}(Q, Q_1)$ *iff*

1. *for each* $A \in \mathbf{d}Q : A \Vdash \mathsf{d}Q_1$, *and*

2. *for each* $B \in \mathbf{d}Q_1$ *there exists a non-empty proper subset* Y *of* $\mathbf{d}Q$ *such that* $B \Vdash Y$.

Here is an example. The question:

What is the breed of this cat: Bombay, European Shorthair, or some other? (7)

implies the question:

Is it a Bombay cat? (8)

Clearly, $\mathbf{Im}^{\odot}(Q, Q_1)$ holds iff $\mathbf{Im}(Q, \emptyset, Q_1)$ is the case.

[11] It is not excluded – but also not required – that direct answers to an implied question are paired with singleton sets of direct answers to the implying question. In such a case we speak about *regular* erotetic implication.

4.2.1 Erotetic Implication and Validity

The premises of an erotetic inference of the second kind comprise a question and, possibly, declaratives, while the conclusion is a question. As long as erotetic inferences of the second kind are considered, IEL proposes the following criteria of validity:

($\mathbf{C}_3$) (Transmission of soundness/truth into soundness). *If the initial question is sound and all the declarative premises are true, then the question which is the conclusion must be sound.*

($\mathbf{C}_4$) (Open-minded cognitive usefulness). *For each direct answer B to the question which is the conclusion there exists a non-empty proper subset Y of the set of direct answers to the initial question such that the following condition holds*:

- ($\Diamond$) *if B is true and all the declarative premises are true, then at least one direct answer $A \in Y$ to the initial question must be true.*

For a throughout discussion, see [44], Chapter 5.

A moment's reflection reveals that the first clause of Definition 5 of erotetic implication expresses in exact terms the idea that lies behind condition ($\mathbf{C}_3$). The same holds true for the second clause of Definition 5 and condition ($\mathbf{C}_4$).

Taken syntactically, an erotetic inference of the second kind is an ordered triple $\langle Q, X, Q_1\rangle$, where Q is the question-premise, X is the set of declarative premises, and Q_1 is the question-conclusion. As before, assume that Q and Q_1, as well as the elements of X, are expressions of a formal language for which erotetic implication has been defined. We put:

Definition 7. *An erotetic inference $\langle Q, X, Q_1\rangle$ is valid iff* $\mathbf{Im}(Q, X, Q_1)$.

The status of Definition 7 resembles that of Definition 4. Comments on the latter (cf. subsection 4.1.1) apply, *mutatis mutandis*, also to the former.

Remark 9. Condition ($\mathbf{C}_4$) is worded semantically, but labelled in pragmatic terms. Note, however, that these are the semantic links between question-conclusion and the respective premises that make the question-conclusion cognitively useful. Suppose that the declarative premises are all true. Since each direct answer to the question-conclusion potentially decreases the class of "options" offered by the question-premise, a true direct answer to the question-conclusion, if found, would actually decrease the class. But recall that each direct answer to the question-conclusion has a disposition to act that way. This is why we speak about "open-minded" cognitive usefulness.

Remark 10. Condition ($\mathbf{C}_4$) and its formal counterpart, the second clause of Definition 5, are demanding, since *every* direct answer to the question-conclusion/implied question is required to possess the disposition mentioned above. One may argue that this is too much and that only some of them should do. Without going into details, let me only mention that experiments and corpora studies have shown that transitions to (auxiliary) questions which are useful in the open-minded way occur quite often (cf. [25] and [22]). Thus erotetic inferences being valid in the sense of Definition 7 are not artefacts.

4.3 Question-Evoking Rules and Question-Implying Rules

Once all the details, syntactic and semantic, of a formal language enriched with questions, are fixed, we are able to move from the semantic to the syntactic level. More precisely, we are able to show what questions (of a formal language considered) are evoked by what sets of d-wffs (of the language), and similarly for erotetic implication. "What" means here "of what syntactic form", since e-wffs are syntactic entities.

For instance, by using a proof method for FOL and a construction of countermodels, one can prove that the following (A, B, C are here metalanguage variables which vary over *atomic* sentences of a first-order language):

$$\mathbf{E}(\{\mathsf{A} \land \mathsf{B} \to \mathsf{C}, \neg\mathsf{C}\}, ?\{\neg\mathsf{A}, \neg\mathsf{B}\}) \tag{9}$$

holds provided that $\mathsf{C} \notin \{\mathsf{A}, \mathsf{B}\}$.[12] Observe that it is a metalogical statement. (9) together with the proviso may be used as the basis for the corresponding *question-evoking rule*, schematically displayed as follows:

$$\frac{\begin{array}{c}\mathsf{A} \land \mathsf{B} \to \mathsf{C} \\ \neg\mathsf{C}\end{array}}{?\{\neg\mathsf{A}, \neg\mathsf{B}\}} \quad \textit{provided} \ \ \mathsf{C} \notin \{\mathsf{A}, \mathsf{B}\} \tag{10}$$

Similarly, one can prove at the metalogical level that the following

$$\mathbf{Im}(?\{A, B, C\}, \{D \to A \lor B, \neg D \to C\}, ?\{D, \neg D\}) \tag{11}$$

is the case for any FOL-sentences A, B, C, D.[13] This leads to the following *question-implying rule*:

$$\frac{\begin{array}{c}?\{A, B, C\} \\ D \to A \lor B \\ \neg D \to C\end{array}}{?\{D, \neg D\}} \tag{12}$$

[12]We apply here the symbolism for questions described in section 3.1. The proviso amounts to the claim that $\mathsf{A}, \mathsf{B}, \mathsf{C}$ are pairwise syntactically distinct, as $\neg\mathsf{A}$ and $\neg\mathsf{B}$ are direct answers. Notice that it is important that A, B, C are supposed to be *atomic* sentences only; for obvious reasons, one cannot generalize 9 to all FOL-sentences.

[13]Since $?\{A, B, C\}$ is a question, A, B, C are supposed to be pairwise syntactically distinct. This time one does not have to restrict oneself to atomic sentences.

Remark 11. There are question-evoking rules that share premises, but not conclusions, and similarly for question-implying rules. For example, besides (9) we also have (under the same proviso):

$$\mathbf{E}(\{\mathsf{A} \wedge \mathsf{B} \to \mathsf{C}, \neg \mathsf{C}\}, ?\{\neg \mathsf{A} \wedge \mathsf{B}, \mathsf{A} \wedge \neg \mathsf{B}, \neg \mathsf{A} \wedge \neg \mathsf{B}\}) \tag{13}$$

and the corresponding question-evoking rule:

$$\frac{\begin{array}{c}\mathsf{A} \wedge \mathsf{B} \to \mathsf{C} \\ \neg \mathsf{C}\end{array}}{?\{\neg \mathsf{A} \wedge \mathsf{B}, \mathsf{A} \wedge \neg \mathsf{B}, \neg \mathsf{A} \wedge \neg \mathsf{B}\}} \ \textit{provided} \ \mathsf{C} \notin \{\mathsf{A}, \mathsf{B}\} \tag{14}$$

Thus one can pass from $\mathsf{A} \wedge \mathsf{B} \to \mathsf{C}$ and $\neg \mathsf{C}$ to $?\{\neg \mathsf{A}, \neg \mathsf{B}\}$ or to $?\{\neg \mathsf{A} \wedge \mathsf{B}, \mathsf{A} \wedge \neg \mathsf{B}, \neg \mathsf{A} \wedge \neg \mathsf{B}\}$, in both cases performing a valid erotetic inference. In this respect IEL is neither worse nor better than Classical Logic and most of its non-classical cousins. For instance, both B and $\neg(B \to A)$ are conclusions of classically valid inferences whose premises comprise $A \vee B$ and $\neg A$.

5 IEL vs. Question Asking and Question Posing

From now on, I will be using the expression *interrogative rules* as a cover term for question-evoking rules and question-implying rules.

5.1 Interrogative Rules vs. Question Asking and Question Posing

Interrogative rules, just like other logical rules, can be characterized set-theoretically. However, let me skip this issue here and concentrate upon their cognitive status.

First, and foremost: interrogative rules *are not* rules which enable questions be proven. Questions as such can not be proven in any reasonable sense of the word "proof." IEL does not aim at proving questions.

Second, an agent who performs a valid erotetic inference (valid in the sense explicated above) need not be aware of the interrogative rule which lies behind the inference. Interrogative rules do not function as premises of erotetic inferences. Also, it is not the case that in order to perform a valid erotetic inference an agent has to to "calculate" the relevant rule(s) first.

Third, IEL differentiates between *question asking* and *question posing*. The crucial difference between them lies in the fact that a posed question, in contradistinction to an asked question, need not be uttered. In order to pose a question one has to ask the question to oneself. One may then ask an interlocutor the question, but it need not be externalised in this way. When looking for a (justified) answer to a posed question, we can attempt to find it on the basis of what we already know or believe, but we may also turn to an external source of information (a literature of the subject matter, a database, etc.) as well as to ask some interlocutor(s). Moreover, it is not always the case that a question asked and the question posed are identical. Questions asked by examiners or by crime investigators constitute classic examples here. Another feature that differentiates question asking from question posing is this: when we ask an interlocutor a question, we usually believe that he/she knows a satisfactory answer or is able to find such an answer. When we pose a question, we are not always convinced that we or available interlocutors are capable to answer the question. It happens that we pose questions of which we know or believe (rightly or not) that we and available interlocutors cannot manage to find satisfactory answers. Last but not least, sometimes questions are asked but not posed. Questions asked for courteous reasons only provide simple examples here.

Interrogative rules are rules of posing questions having some desired properties with respect to previously accepted (maybe only hypothetically) declarative premises and/or previously posed questions. Question-evoking rules pave the way for arriving at questions which are sound if the premises used are true, and which are informative relative to the premises. Question-implying rules, in turn, facilitate arriving at questions which are sound relative to questions initially posed and the premises used, and which are cognitively useful in the sense explicated by the condition ($\mathbf{C}_4$) above. This is not much, but still something.

However, one can expect more from a *logic* of questions. It should give an account of what questions *are to be* asked in a given cognitive situation. Moreover, it should shed light on question answering. IEL addresses these issues and proposes some solutions. But interrogative rules are not keys to the solutions offered.

6 The Decomposition Issue

Questions and questioning are closely intertwined with problem solving. One of the crucial principles which govern effective problem solving is the following:

(DP) (Decomposition principle): *Decompose a principal problem (PP) into simpler sub-problems (SPs) in such a way that solutions to SPs can be assembled into an overall solution to PP.*

When we are concerned with a problem definite enough to be adequately expressed by a question, its decomposition amounts, generally speaking, to finding an appropriate collection of auxiliary questions. A decomposition can be *static*, that is, resulting in finding a set of mutually independent auxiliary questions such that once *all* of them are answered, the initial problem is resolved. Yet, a more interesting case is that of *dynamic* decomposition that comes in *stages*: the consecutive auxiliary questions (which constitute the sub-goals of the next stage) depend on how the previous requests for information have been fulfilled. The main goal, determined by the principal problem, remains unchanged, but sub-goals are processed in a goal-directed way. Moreover, the erotetic decomposition principle:

(EDP) (Erotetic decomposition principle): *Transform a principal question into auxiliary questions in such a way that: (a) consecutive auxiliary questions are dependent upon previous questions and, possibly, answers to previous auxiliary questions, and (b) once auxiliary questions are resolved, the principal question is resolved as well.*

is observed.

IEL models static decomposition by using a semantic concept of *reducibility of a question to a set of questions* (cf. [37]). In particular, many feasibility results have been proven (cf., e.g., [38], pp. 194–200, or [13]). As for dynamic decomposition, a basic tool of analysis is the concept of *erotetic search scenario*, introduced in [40].

6.1 Erotetic Search Scenarios

Erotetic search scenarios (e-scenarios for short) are abstract entities. Let Q be a question and X be a (possibly empty) set of d-wffs. An e-scenario for Q relative to X can be defined either as a family of interconnected sequences of questions and d-wffs, or as a finite labelled tree, where the labels are questions and d-wffs. There is no room for presenting exact definitions here, so I will provide only an informal description based on the labelled trees approach.

The root of an e-scenario for question Q relative to a set of d-wffs X is labelled by question Q, being the *principal question* of the e-scenario. The leaves are labelled by direct answers to the principal question. Nodes of an e-scenario are labelled by questions or by d-wffs. For brevity, let us call the former e-nodes and the latter d-nodes. Questions labelling e-nodes different from the root – *auxiliary questions* of e-scenarios – enter them due to erotetic implication.[14] To be more precise, it is requested that each auxiliary question of a branch (i.e. a maximal path) must be erotetically implied by some question which labels a preceding node of the branch, the principal question included, possibly on the basis of some d-wff(s) which label preceding node(s) of the branch. An immediate successor of an e-node different from the root is labelled either by a question or by a d-wff. In the latter case it is required that the d-wff is a direct answer to the auxiliary question which labels the node. Moreover, it is requested that

[14] This is the main feature that distinguishes e-scenarios form epistemic erotetic search scenarios (cf. [24]), in which relations between questions are determined by epistemic erotetic logic of Peliš (cf. [30]).

each direct answer to the question labels some immediate successor of the e-node. If, however, an immediate successor of an e-node is an e-node, it the only immediate successor of the e-node. An auxiliary question of an e-scenario that labels a node whose immediate successors are labelled by direct answers to the question is a *query* of the scenario. Note that an e-scenario may involve auxiliary questions that are not queries. Each d-node is supposed to have at most one immediate successor. A d-wff which labels a d-node of a branch must fulfil at least one of the following conditions: (a) it belongs to the set X, (b) it is a direct answer to the auxiliary question which labels the preceding node of the branch, or (c) it is entailed by some wff(s) which label preceding node(s) of the branch. Observe that it is neither assumed nor denied that each d-wff in X labels some node. These which do are *declarative premises* of an e-scenario. Finally, it is requested that no direct answer to the principal question Q belongs to the set X, and that no auxiliary question is set-theoretically equivalent to Q, that is, the set of direct answers to it equals the set of direct answers to Q. Figures 1 and 2 display examples of e-scenarios. (To enhance readability, e-wffs of the form $?\{A \wedge B, A \wedge \neg B, \neg A \wedge B, \neg A \wedge \neg B\}$ are abbreviated as $? \pm |A, B|$.)

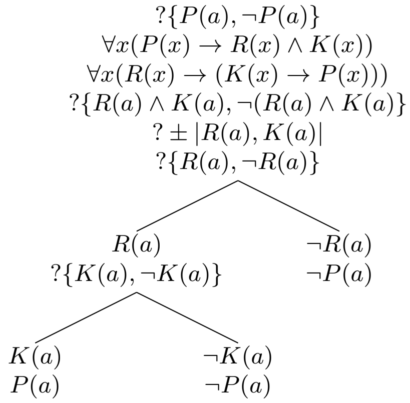

Figure 1: *An example of e-scenario for* $?\{P(a), \neg P(a)\}$ *relative to the set of d-wffs* $\{\forall x(P(x) \to R(x) \wedge K(x)), \forall x(R(x) \to (K(x) \to P(x)))\}$. ($P$, Q, K are one-place predicates, while a stands for an individual constant.)

One can prove that if the principal question of an e-scenario is sound and all the declarative premises ot the scenario are true, then the e-scenario has at least one "golden path" i.e. a branch whose nodes are labelled by sound questions and true d-wffs. As leaves of an e-scenario are labelled by direct answers to the principal question, a "golden path" leads to a true direct answer to the question.

IEL defines some operations on e-scenarios (cf. [44], Chapter 11, and [5]), which produce e-scenarios from e-scenarios and enable their optimization. This makes possible an automation of e-scenarios generation (cf. [5]).

$$
\begin{array}{c}
?\{P(a), \neg P(a)\} \\
\forall x(P(x) \leftrightarrow R(x) \vee (K(x) \wedge H(x))) \\
? \pm |R(a), K(a) \wedge H(a)| \\
?\{R(a), \neg R(a)\}
\end{array}
$$

- $R(a)$
  $P(a)$
- $\neg R(a)$
  $?\{K(a) \wedge H(a), \neg(K(a) \wedge H(a))\}$
  $?\{K(a) \wedge H(a), K(a) \wedge \neg H(a), \neg K(a)\}$
  $?\{K(a), \neg K(a)\}$
  - $K(a)$
    $?\{H(a), \neg H(a)\}$
    - $K(a)$
      $K(a) \wedge H(a)$
      $P(a)$
    - $\neg K(a)$
      $\neg(K(a) \wedge H(a))$
      $\neg P(a)$
  - $\neg K(a)$
    $\neg(K(a) \wedge H(a))$
    $\neg P(a)$

Figure 2: *An example of e-scenario for* $?\{P(a), \neg P(a)\}$ *relative to* $\{\forall x(P(x) \leftrightarrow R(x) \vee (K(x) \wedge H(x))\}$. ($P, R, K, H$ are one place predicates and a stands for an individual constant.)

Looking from the pragmatic point of view, an e-scenario for a question Q shows what other questions are potentially worth to be asked in order to answer the question Q. Moreover, it provides us with instructions as to when they are advised to be asked. These instructions pertain to queries. Queries of e-scenarios are carriers of information requests. An e-scenario shows what is the first advisable query, and what is the next advisable query if the information request of a previous query has been satisfied in such-and-such way. The provided instructions are conditional: if one receives answer A_1 to query Q^*, query Q_1^* should be asked next. If, however, one receives answer A_2 to Q^*, question Q_2^* is the next recommended query, etc. What is important, an e-scenario does this with regard to any possible way of satisfying the request of a query, where the ways are determined by direct answers to the question which expresses the query. Moreover, an e-scenario behaves in this manner in the case of each query of the e-scenario. Thus the e-scenarios approach transcends the common schema of "production of a sequence of questions and affirmations." The fact that information requests can be satisfied in one way or another is treated seriously: for any query and any possible way of satisfying the request carried by it there is an instruction concerning "what to do next."

The execution of an e-scenario proceeds from top to bottom: one attempts to resolve the first query and then, depending on the answer received, moves to the query recommended by the e-scenario as the next one, and so forth until there is no further query. When an e-scenario is executed, instruction based on answers different from those actually got will not be activated.

E-scenarios were initially designed as tools which may be useful in formal modelling of problem solving.[15] But the range of applicability of the concept occurred to be wider. It includes question answering, in particular answering with questions (cf. [45]) as well as cooperative answering (cf. [23]), and dialogue modelling in general (cf. [21]). Some applications of the concept in proof theory have also been found.

7 IEL Meets Proof Theory

7.1 From IEL to Proof Theory

The method of synthetic tableaux (cf., e.g. [33], [34] [19]) originates, in a sense, from considerations upon e-scenarios. Another example is provided in [41], where a proof system for Classical Propositional Logic, in which rules transform e-scenarios into e-scenarios, and proofs are conceived as sequences of e-scenarios, is presented. The philosophical idea that laid behind this, rather specific, proof format, was: in order to prove A, a systematic reflection on possible ways of reaching A which shows that reaching the opposite requires incoherent information, is sufficient. The approach presented in [41] has not been generalised to other logics, however.

The philosophical idea that lies behind another proof method grounded in IEL, the *method of Socratic proofs*, is different. There are problems which can be solved by pure questioning, that is, by transforming the relevant initial question into consecutive questions until a question which, for obvious reasons, can be rationally answered in only one way, is arrived at. Once this is achieved, a *Socratic proof* of a solution is found. The method of Socratic proofs gives an account of this idea in regard to logical problems concerning, for instance, entailment/derivability, validity/theoremhood, or inconsistency. How is it done? Erotetic calculi are proposed. A calculus of this kind consists of rules which transform questions into questions. There are no axioms. Instead, questions which, if arrived at, turn a transformation into a Socratic proof, are characterized in syntactic terms. An erotetic calculus is grounded in IEL, as it operates with rules which are question-implying rules. Since further general explanations would rather multiply doubts than dissolve them, let me give an example. The erotetic calculus briefly presented below deals with negation-implication fragment of Classical Propositional Logic. We label this calculus with $\mathbb{E}^{\mathsf{CPL}}_{\neg,\rightarrow}$.

[15]There is no room for showing how the concept has been applied there. Let me only mention that the simple scheme: “first design a scenario, and then execute it” is not the only one used. An interested reader is advised to consult [44], Chapter 13, for details.

7.1.1 The Erotetic Calculus $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$

For brevity, I will be more formal in this section than in the previous ones.

Let $\mathcal{L}$ be the language of Classical Propositional Logic (henceforth: CPL) with negation, $\neg$, and implication, $\to$, as the only primitive connectives. Wffs of $\mathcal{L}$ are defined in the standard manner. The semantics of $\mathcal{L}$ is the usual one. A CPL-*valuation* is a mapping v of the set of wffs of $\mathcal{L}$ into the set of logical values, $\{\mathbf{1}, \mathbf{0}\}$, such that for any wffs A, B of $\mathcal{L}$: (a) $v(\neg A) = \mathbf{1}$ iff $v(A) = \mathbf{0}$, and (b) $v(A \to B) = \mathbf{1}$ iff $v(a) = \mathbf{0}$ or $v(B) = \mathbf{1}$. A set of wffs X of $\mathcal{L}$ *entails* a wff A of the language, in symbols $X \models A$, just in case there is no CPL-valuation that assigns $\mathbf{1}$ to all elements of X and assigns $\mathbf{0}$ to A.

Now let us consider expressions of the form:

$$S \vdash A \tag{15}$$

where S is a (possibly empty) finite sequence of wffs of $\mathcal{L}$, and A is a wff of $\mathcal{L}$. Call them single-conclusioned sequents or *sequents* for short. The turnstile $\vdash$ does not occur in the vocabulary of $\mathcal{L}$, so sequents are not wffs of $\mathcal{L}$. However, they can be evaluated in terms of semantics of $\mathcal{L}$. Let $[S]$ stand for the set of all the wffs of $\mathcal{L}$ which are terms of the sequence S.[16] We say that sequent $S \vdash A$ is CPL-*valid* if $[S]$ entails A.

Among sequents, the basic ones play a distinguished role. A sequent $S \vdash A$ is *basic* iff (a) A is a term of S, or (b) there exists a wff C such that both C and $\neg C$ are terms of S. Clearly, each basic sequent is CPL-valid, but not the other way round.

Rules of the erotetic calculus $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$ operate on expressions of a language $\mathcal{L}^*$, which is built on top of $\mathcal{L}$.

The vocabulary of $\mathcal{L}^*$ includes the vocabulary of $\mathcal{L}$, and the following signs: $\vdash$, ?, $\sim$, &. Sequents are *atomic* d-wffs of $\mathcal{L}^*$. The set $\mathcal{D}$ of *d-wffs* of $\mathcal{L}^*$ is the smallest set that includes all atomic d-wffs of the language and fulfils the following conditions: (a) if $\mathfrak{u} \in \mathcal{D}$, then '$\sim \mathfrak{u}$' $\in \mathcal{D}$; (b) if $\mathfrak{u}, \mathfrak{r} \in \mathcal{D}$, then '$(\mathfrak{u} \,\&\, \mathfrak{r})$' $\in \mathcal{D}$.

Questions (i.e. e-wffs) of $\mathcal{L}^*$ are of the form:

$$?(S_1 \vdash A_1, \ldots, S_k \vdash A_k) \tag{16}$$

where $S_1 \vdash A_1, \ldots, S_k \vdash A_k$ ($k \geqslant 1$) is a finite sequence of *atomic* d-wffs of $\mathcal{L}^*$, that is, of sequents. Each term of the sequence is called a *constituent* of the question. The set of direct answers to (16) comprises the *affirmative answer*:

$$S_1 \vdash A_1 \,\&\ldots\&\, S_k \vdash A_k \tag{17}$$

and the *negative answer*:

$$\sim (S_1 \vdash A_1 \,\&\ldots\&\, S_k \vdash A_k) \tag{18}$$

[16] We need this technical notion, since it is neither assumed nor denied that S is a sequence without repetitions.

The intuitive meaning of a question of the form (16) is:

Is it the case that: $[S_1]$ *entails* A_1 *and* ... *and* $[S_k]$ *entails*A_k?

If a question has only one constituent, i.e. of the form:

$$?(S \vdash A) \tag{19}$$

its intuitive meaning is:

Does $[S]$ *entail* A?

Questions of $\mathcal{L}^*$ are expressions of a language built on top of $\mathcal{L}$. They concern entailment in $\mathcal{L}$, however. The syntax of $\mathcal{L}^*$ is well-specified and thus $\mathcal{L}^*$ itself is an object-level formal language, analogously as $\mathcal{L}$ is.

$\mathcal{L}^*$ is supplemented with its own semantics. It is based on the concept of admissible partition.[17] A *partition* of $\mathcal{D}$ (i.e. of the set of d-wffs of $\mathcal{L}^*$) is an ordered pair:

$$\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle \tag{20}$$

such that $\mathcal{D} = \mathsf{T_P} \cup \mathsf{U_P}$ and $\mathsf{T_p} \cap \mathsf{U_P} = \emptyset$. A partition $\langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of $\mathcal{D}$ is *admissible* if it fulfils the following conditions:[18]

1. $\ulcorner S\,'\,A \to B\,'\,T \vdash C \urcorner \in \mathsf{T_P}$ iff $\ulcorner S\,'\,\neg A\,'\,T \vdash C \urcorner \in \mathsf{T_P}$ and $\ulcorner S\,'\,B\,'\,T \vdash C \urcorner \in \mathsf{T_P}$;
2. $\ulcorner S\,'\,\neg(A \to B)\,'\,T \vdash C \urcorner \in \mathsf{T_P}$ iff $\ulcorner S\,'\,A\,'\,\neg B\,'\,T \vdash C \urcorner \in \mathsf{T_P}$;
3. $\ulcorner S\,'\,\neg\neg A\,'\,T \vdash B \urcorner \in \mathsf{T_P}$ iff $\ulcorner S\,'\,A\,'\,T \vdash B \urcorner \in \mathsf{T_P}$;
4. $\ulcorner S\,'\,T \vdash A \to B \urcorner \in \mathsf{T_P}$ iff $\ulcorner S\,'A\,'\,T \vdash B \urcorner \in \mathsf{T_P}$;
5. $\ulcorner S \vdash \neg(A \to B) \urcorner \in \mathsf{T_P}$ iff $\ulcorner S \vdash A \urcorner \in \mathsf{T_P}$ and $\ulcorner S \vdash \neg B \urcorner \in \mathsf{T_P}$;
6. $\ulcorner S \vdash \neg\neg A \urcorner \in \mathsf{T_P}$ iff $\ulcorner S \vdash A \urcorner \in \mathsf{T_P}$;
7. $\ulcorner (\mathfrak{u}\ \&\ \mathfrak{r}) \urcorner \in \mathsf{T_P}$ iff $\mathfrak{u} \in \mathsf{T_P}$ and $\mathfrak{u} \in \mathsf{T_P}$;
8. $\ulcorner \sim \mathfrak{u} \urcorner \in \mathsf{T_P}$ iff $\mathfrak{u} \notin \mathsf{T_P}$.

Note that the above conditions are not *ad hoc*. Conditions 1 - 6 reflect the behaviour of implications, negated implications, and double negated wffs in the context of entailment, while conditions 7 and 8 show that the $\mathcal{L}^*$-negation,$\sim$, and the $\mathcal{L}^*$-conjunction, &, are classical.

A d-wff $\mathfrak{u}$ of $\mathcal{L}^*$ *entails* a d-wff $\mathfrak{r}$ of $\mathcal{L}^*$ iff there is no admissible partition $\mathsf{T_P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ of the set $\mathcal{D}$ of d-wffs of $\mathcal{L}^*$ such that $\mathfrak{u} \in \mathsf{T_P}$ and $\mathfrak{r} \in \mathsf{U_P}$. Notice that this time we speak about entailment between d-wffs of $\mathcal{L}^*$, further on referred to as entailment *in* $\mathcal{L}^*$.

[17]Semantics of this kind are commonly used in IEL; cf. [44], Chapter 3.

[18]The symbol $'$ stands for the concatenation-sign for sequences of wffs of $\mathcal{L}$. Thus $S\,'\,T$ is the concatenation of a sequence of wffs S and a sequence of wffs T. An expression of the form $S\,'\,A$ represents the concatenation of S and the one-term sequence whose term is A. Of course, $S\,'\,A\,'\,T$ is the concatenation of $S\,'\,A$ and T. The letters $\mathfrak{r}$, $\mathfrak{u}$ are metalanguage variables for d-wffs of $\mathcal{L}^*$.

The erotetic calculus $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$ has no axioms, but comprises the following rules:[19]

$$\mathsf{L}_{\to} \quad \frac{?(\Phi; S'A \to B'T \vdash C; \Psi)}{?(\Phi; S'\neg A'T \vdash C; S'B'T \vdash C; \Psi)} \qquad \mathsf{R}_{\to} \quad \frac{?(\Phi; S \vdash A \to B; \Psi)}{?(\Phi; S'A \vdash B; \Psi)}$$

$$\mathsf{L}_{\neg\to} \quad \frac{?(\Phi; S'\neg(A \to B)'T \vdash C; \Psi)}{?(\Phi; S'A'\neg B'T \vdash C; \Psi)} \qquad \mathsf{R}_{\neg\to} \quad \frac{?(\Phi; S \vdash \neg(A \to B); \Psi)}{?(\Phi; S \vdash A; S \vdash \neg B; \Psi)}$$

$$\mathsf{L}_{\neg\neg} \quad \frac{?(\Phi; S'\neg\neg A'T \vdash C; \Psi)}{?(\Phi; S'A'T \vdash C; \Psi)} \qquad \mathsf{R}_{\neg\neg} \quad \frac{?(\Phi; S \vdash \neg\neg A; \Psi)}{?(\Phi; S \vdash A; \Psi)}$$

A rule acts upon a constituent of a question with regard to an occurrence of a wff of $\mathcal{L}$ in the constituent. The resultant question differs from the initial question only in having one or two new constituents at the place where the initial question had the constituent affected. With the exception of rule $\mathsf{R}_{\to}$, a new constituent differs from the constituent acted upon only in having a new wff or wffs at the place of the wff acted upon. As for $\mathsf{R}_{\to}$, one new wff occurs just left of the turnstile, while the other replaces the implication acted upon. Side constituents of the constituent acted upon, Φ and Ψ, if non-empty, are transferred to the resultant question without changing their order.

One can prove that each of the above rules ensures erotetic implication and thus is a question-implying rule. To see this, it suffices to observe that if question Q_1 results from question Q by a rule of $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$, then the affirmative answers to Q_1 and Q entail (in $\mathcal{L}^*$) each other, and the negative answers to Q_1 and Q entail (again, in $\mathcal{L}^*$) each other.

Rules of $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$ enable the so-called Socratic transformations of questions of $\mathcal{L}^*$. A *Socratic transformation* of a question Q *via* the rules of $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$ is a sequence of questions $Q_1, Q_2, \ldots$ such that $Q_1 = Q$ and for each $i \geqslant 1$, Q_{i+1} results from Q_i by a rule of $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$.

Given what has been said above, the following comes with no surprise:

($\heartsuit$) *Each step of a Socratic transformation, i.e. a transition from a question to the next one, is a valid erotetic inference.*

In particular, this pertains to the so-called successful Socratic transformations. A Socratic transformation of a question Q via the rules of $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$ is *successful* if it is finite and each constituent of the last question of the transformation is a basic sequent.

[19]As before, the letters S, T, U, W stand for finite (possibly empty) sequences of wffs of $\mathcal{L}$, and $'$ is the concatenation-sign for these sequences. The letters Φ, Ψ are metalanguage variables for finite (again, possibly empty) sequences of atomic d-wffs of $\mathcal{L}^*$, and the semicolon is used as the concatenation-sign for these sequences. One-term sequences are represented by their terms.

Viewed in the perspective of semantics of the "initial" language $\mathcal{L}$, rules of the calculi $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$ have the following property:

(♠) *If question $Q^\star$ results from question Q by a rule of $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$, then all the constituents of Q are* CPL*-valid iff all the constituents of $Q^\star$ are* CPL*-valid.*

Now recall that basic sequents are CPL-valid. Thus, by (♠), all the constituents of the first question of a successful Socratic transformation are CPL-valid. So when a question which has only one constituent, i.e. is of the form:

$$?(S \vdash A)$$

happens to be Socratically transformed (via the rules of $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$) *with a success*, the sequent $S \vdash A$ is CPL-valid! One does not need any further calculations to establish its CPL-validity. It is established or "proven" by performing a series of valid erotetic inferences, starting with an inference whose premiss is a yes-no question about CPL-validity of the sequent. Now recall that the last question of a successful Socratic transformation asks whether all the basic sequents involved are CPL-valid. But each basic sequent is CPL-valid due to general properties of entailment: any wff is entailed by a set of wffs which contains the wff, and any wff is entailed by a set of wffs which contains contradictory wffs. In this sense the last question is a "rhetorical" one.

All what has been said above leads to the concept of Socratic proof. A *Socratic proof* of sequent $S \vdash A$ in the erotetic calculus $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$ is a successful Socratic transformation of the question $?(S \vdash A)$ via the rules of $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$. Notice that Socratic proofs are not proofs of questions, but proofs of sequents.

Here is an example of a Socratic proof of the sequent:

$$p \to (q \to r), q \vdash (p \to r)$$

1. $?(p \to (q \to r), q \vdash (p \to r))$
2. $?(p \to (q \to r), q, p \vdash r))$
3. $?(\neg p, q, p \vdash r; q \to r, q, p \vdash r)$
4. $?(\neg p, q, p \vdash r; \neg q, q, p \vdash r; r, q, p \vdash r)$

One can show that each CPL-valid sequent made up of wffs of $\mathcal{L}$ is provable in $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$.

7.1.2 Other Erotetic Calculi

The erotetic calculus $\mathbb{E}^{\mathsf{CPL}}_{\neg,\to}$ is a toy example. There exist erotetic calculi for the full CPL (cf. [42]), First-Order Logic (cf. [49], [3]), and for some non-classical logics. In particular, intuitionistic propositional logic (cf. [18]) and normal modal propositional logics have been dealt with (cf. [14], [18], [15], [16]), as well as some paraconsistent logics and logics of formal inconsistency (cf. [50], [6]). Needless to say, erotetic calculi dealing with logics other than mere negation-implication fragment of CPL have more complicated setups, both on the syntactic and the semantic level.

One can argue that the existence of erotetic calculi is nothing but a curiosity. A general philosopher might have replied with saying that their existence reveals the priority of questioning over answering. An analytic philosopher might have added that the existence of erotetic calculi sheds new light on analyticity of logic. A logician working on proof-search issues may find it interesting that proofs in the erotetic calculi format can be transformed into proofs in sequent calculi (cf. [20]), proofs in the analytic tableaux format (cf. [17]), and Hilbert-style proofs (cf. [10]). Besides this, the method of Socratic proofs contributes to proof theory in other ways as well. A reader interested in details is strongly advised to consult the monograph [18]. Last but not least, the method of Socratic proofs has been applied in formal modelling of abductive reasoning (cf. [32], [4]).

7.2 Towards Proof-theoretic Accounts of Question Evocation and Erotetic Implication

The basic concepts of IEL, question evocation and erotetic implication, are semantic. As illustrated in subsection 4.3, one can move from the semantic level to the syntactic level by showing what questions are erotetically implied/evoked by what sets of wffs and/or questions, where "what" means "of what syntactic form." As it has been shown in [48], in many cases it is possible to "extract" multiple conclusion-entailment, the basic semantic concept by means of which question evocation and erotetic implication are defined, from the consequence relation of the underlying logic of d-wffs. This, in a sense, grounds IEL in proof theory. However, one may argue as follows. Since validity of erotetic inferences is defined in terms of question evocation and erotetic implication, these concepts function in IEL analogously to the concept of entailment in other logics. It is natural to expect a proof-theoretic account of entailment. So one may expect the same for question evocation as well as for erotetic implication. Until now, there exist only a few logical calculi in which, generally speaking, formulas expressing question evocation, erotetic implication, or both, become provable (cf. [35], [27], [9], [8], [47], [28], [29], [7]). These calculi differ in many respects. I will not comment here on their pros and cons. However, since work on the subject is, as a matter of fact, in an early stage, let me end this essay with some remarks which, I hope, clarify what "providing a proof-theoretic account of question evocation and/or erotetic implication" aims at.

Let $\mathfrak{L}$ be an arbitrary but fixed formal language enriched with questions (that is, a language of the kind described in subsection 3.1), in which the set of d-wffs and the set of e-wffs are disjoint. $\mathfrak{L}$ is an object-level formal language. Assume that the language is supplemented with a semantics rich enough to define the concept of truth for d-wffs and the relation of multiple-conclusion entailment between sets of d-wffs of the language. This allows us to define question evocation and erotetic implication. However, they are semantic relations between d-wffs and e-wffs of $\mathfrak{L}$ defined on the metalanguage level. An object level formal language usually lacks formulas by means of which relations defined in this way are directly expressed. A solution is to build a second formal language, say, $\mathfrak{L}^{\circ}$, being an extension of $\mathfrak{L}$. One can build such a language in a very simple manner. We extend the vocabulary of $\mathfrak{L}$ with a sign, $\Rightarrow$. The choice of $\Rightarrow$ is arbitrary; any other sign can do. A reader is advised to suspend any associations he/she may have. Besides $\Rightarrow$, we also need in $\mathfrak{L}^{\circ}$ expressions which refer to sequences of d-wffs of $\mathfrak{L}$ or to sets (possibly multisets of d-wffs of $\mathfrak{L}$. When we restrict ourselves to finite sets or sequences, this can be achieved relatively easy, by allowing lists of d-wffs of $\mathcal{L}$ to be constituents of well-formed formulas of $\mathfrak{L}^{\circ}$. In what follows I assume that expressions referring to sequences/sets of d-wffs of $\mathfrak{L}$ occur in $\mathfrak{L}^{\circ}$. I will be using the letters Σ, Γ as metalanguage variables for such expressions. $Q, Q_1, \ldots$ are supposed to vary over e-wffs of $\mathfrak{L}$.

Well-formed formulas (wffs) of $\mathfrak{L}^{\circ}$ fall into the schemata:

$$\Sigma \Rightarrow Q \tag{21}$$

$$Q, \Sigma \Rightarrow Q_1 \tag{22}$$

$$\Sigma \Rightarrow \Gamma \tag{23}$$

Let us now consider the following structure:

$$\langle \mathfrak{D} \cup \mathfrak{E}, \mathbf{d}, \|\!\!=, \mathsf{E}, \mathsf{Im} \rangle \tag{24}$$

where $\mathfrak{D}$ is the set of d-wffs of $\mathfrak{L}$, $\mathfrak{E}$ is the set of e-wffs of $\mathfrak{L}$, and $\mathbf{d}$ is a (possibly partial) function from $\mathfrak{E}$ to $\wp(\mathfrak{D})$. Intuitively, $\mathbf{d}$ is the answerhood function: $\mathbf{d}Q$ constitutes the set of direct answers to Q provided that Q belongs to the set of arguments of $\mathbf{d}$. The remaining items, $\|\!\!=$, E, and Im, are multiple-conclusion entailment in $\mathfrak{L}$, question evocation in $\mathfrak{L}$, and erotetic implication in $\mathfrak{L}$, respectively.[20] The structure (24) is the *intended model*

[20]These relations are construed here set-theoretically. Question evocation *in* $\mathfrak{L}$ is the subset of the Cartesian product of $\wp(\mathfrak{D})$ and $\mathfrak{E}$ such that for each ordered pair $\langle X, Q \rangle$ belonging to the subset, $\mathbf{E}(X, Q)$ holds. Similarly for erotetic implication in $\mathfrak{L}$, and mc-entailment in $\mathfrak{L}$.

for $\mathfrak{L}^\circ$. Let us designate it by $\mathfrak{M}^\circ$. The truth conditions are ('$\mathfrak{M}^\circ \models \mathfrak{G}$' abbreviates '$\mathfrak{G}$ is true in $\mathfrak{M}^\circ$'):

1. $\mathfrak{M}^\circ \models \Sigma \Rightarrow Q$ *iff* $\mathbf{E}(|\Sigma|, Q)$.
2. $\mathfrak{M}^\circ \models Q, \Sigma \Rightarrow Q_1$ *iff* $\mathbf{Im}(Q, |\Sigma|, Q_1)$.
3. $\mathfrak{M}^\circ \models \Sigma \Rightarrow \Gamma$ *iff* $|\Sigma| \Vdash |\Gamma|$.

When Σ is a sequence, $|\Sigma|$ is the set of all terms of the sequence. If Σ is a multiset, $|\Sigma|$ stand for the set of all its elements. If Σ is a set, then $|\Sigma| = \Sigma$.

We need two auxiliary notions.

Definition 8. $\mathbf{Th}(\mathfrak{M}^\circ) =_{df} \{\mathfrak{G} : \mathfrak{M}^\circ \models \mathfrak{G}\}$.

$\mathbf{Th}(\mathfrak{M}^\circ)$ is thus the set of all the wffs of $\mathfrak{L}^\circ$ that are true in the intended model for the language.

Definition 9.

1. $\mathbf{T_{qe}} = \{\mathfrak{G} \in \mathbf{Th}(\mathfrak{M}^\circ) : \mathfrak{G}$ *is of the form* $\Sigma \Rightarrow Q\}$.
2. $\mathbf{T_{ei}} = \{\mathfrak{G} \in \mathbf{Th}(\mathfrak{M}^\circ) : \mathfrak{G}$ *is of the form* $Q, \Sigma \Rightarrow Q_1\}$.
3. $\mathbf{T_{me}} = \{\mathfrak{G} \in \mathbf{Th}(\mathfrak{M}^\circ) : \mathfrak{G}$ *is of the form* $\Sigma \Rightarrow \Gamma\}$.

We are now able to clarify what a proof-theoretic account of question evocation in $\mathfrak{L}$ aims at. A logical calculus accomplishes the task just in case at least some (but ideally all) wffs from $\mathbf{T_{qe}}$ become calculable by means of the calculus.[21] I intentionally use here the term "calculable"[22] instead of "provable", since the former is less loaded than the latter. At this moment I do not want to forejudge the proof format of a calculus. Speaking about becoming calculable presupposes only the existence of rules and, possibly, axioms of some kind or another. Both rules and axioms are supposed to be defined in purely syntactic terms.

One may expect from a logical calculus that provides an account of question evocation being erotetically homogenous, that is, have primary rules which operate only on formulas of the form $\Sigma \Rightarrow Q$ and axioms, if there are any, expressed by such formulas. But no calculus known in the literature is erotetically homogenous. Of course, homogenity of this kind is a matter of elegance only. After all, question evocation is defined in terms of entailment between d-wffs. A calculus whose rules operate, among others, on formulas different from these falling into the schema $\Sigma \Rightarrow Q$, simply reflects this fact. So "mixed" rules, that is, rules involving in their

[21] Note that we do not require all the wffs from $\mathbf{T_{qe}}$ be calculable by means of a calculus. Calculi which "provide" all of them are *complete* w.r.t. the corresponding E. It is already known that in some important cases complete calculi do not exist (cf. [7]).

[22] Borrowed in this context from Cordes [7].

antecedents both schemata of formulas of the form $\Sigma \Rightarrow Q$ and schemata of formulas of the form $\Sigma \Rightarrow \Gamma$ are acceptable, as well as subsidiary rules which do not operate with formulas of the form $\Sigma \Rightarrow Q$ at all. A minimal elegance requirement seems to be: at least one primary rule of a calculus has schemata of formulas of the form $\Sigma \Rightarrow Q$ both in the antecedent and in the consequent. But, again, even this is not mandatory.

Everything what has been said above on providing a proof-theoretic account of question evocation can be repeated, *mutatis mutandis*, in regard to giving such an account of erotetic implication.

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