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KUBIŃSKI'S THEORY OF QUESTIONS

0. INTRODUCTION

Tadeusz Kubiński has made an important contribution to the logic of questions. He was one of the pioneers of applying the methods and techniques of modern formal logic to the field. In the sixties and early seventies Kubiński published several very interesting papers devoted to the logic of questions. His monograph *Wstęp do logicznej teorii pytań* [*An Introduction to the Logical Theory of Questions*] was published in 1971 in Warsaw [21]. This book preceded the famous monograph *The Logic of Questions and Answers* published by Nuel D. Belnap and Thomas P. Steel in 1976 [3] and even now may be regarded as an alternative to it: there are some similarities in the general approach (questions are not reduced to expressions of other kinds, syntactical tools are extensively used, model-theoretic semantics is applied etc.), but there are also substantial differences. Unfortunately Kubiński published his book in Polish; his English monograph based on the Polish book, titled *An Outline of the Logical Theory of Questions*, was published only in 1980 in (East) Berlin [25].

This paper is devoted to the presentation of two important aspects of the Kubiński's contribution to the logic of questions. First, we shall present here his analysis of the logical form of questions. Second, we will present two systems of erotetic logic built by him. Let us stress, however, that our aims are limited: we do not pretend here to the complete exposition of all of Kubiński's ideas and results in the area of the logic of questions. This paper is only an introduction. Yet, we hope that it will encourage both logicians and linguists interested in questions to study Kubiński's papers and monographs in detail.

There is no room for a comparison of Kubiński's approach with other approaches. For a general information and assessment, however, see the review article [6] and the paper of Harrah included in this volume.

1. LOGICAL ANALYSIS OF QUESTIONS

In this section we shall present an outline of Kubiński's analysis of the logical form of questions. In what follows the expressions "question", "interrogative sentence" and "interrogative" will be used as synonyms.

1.1. The concept of question. Kubiński's concept of question of a formalized language is purely syntactical: questions are understood as expressions of a strictly defined shape. The leading idea of the analysis is that each question consists of an *interrogative operator* and a *sentential function.* Interrogative operators, in turn, consist of both constants and variables. The only free variables in the sentential functions which occur in questions are the variables of the corresponding interrogative operators; these variables are "bound" by the interrogative operators. Thus the structure of questions resembles to some extent the structure of some quantified declarative formulas. Yet, the interrogative operators are not defined in terms of quantifiers and questions are not reduced to declarative formulas. Moreover, questions are not reduced to expressions of other kinds, such as imperatives, epistemic imperatives, alethic modalities etc.

The variables which occur in questions may belong to various syntactical categories. Roughly, the categories of variables "bound" by interrogative operators indicate the (ontological) categories of objects which are asked about. For example, a question whose interrogative operator contains only individual variables asks about individuals. If the relevant variables run over sentential connectives, then the corresponding questions are about either the existence of some state(s) of affairs or some connection(s) between states of affairs. Questions with predicate variables, in turn, ask about properties or relations. When a question contains only sentential variables, it is a question about logical values (truth and falsity). Kubiński considers also "mixed" questions, that is, questions whose interrogative operators contain variables belonging to two or more different categories.

The monograph [21] and its enriched English version [25] provide us a formal approach to questions of many kinds. Some of the results of these monographs were announced in the papers of Kubiński published up to 1970 (see References); yet, these monographs also refine some previous proposals. The papers of Kubiński published in 1973 contain, in turn, elements of a new, more general approach. There is no room for the presentation of all the results of these monographs and papers; we shall restrict ourselves to the analysis of questions with individual variables and with variables running over sentential connectives. Roughly, the first may be regarded as formal counterparts of some "which" questions, whereas the second are formal counterparts of some propositional questions.

1.2. Basic formal language. Questions analyzed by Kubiński are expressions of various formalized languages. These languages, however, result from some basic first-order language by extending its vocabulary with some new variables and constants. The basic language $\mathscr{L}$ is simply the language of the first-order predicate calculus with identity and individual constants, but without function symbols. The vocabulary of $\mathscr{L}$ contains the logical constants $\neg$ (negation), & (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\equiv$ (equivalence), $\forall$ (universal quantifier), $\exists$ (existential quantifier), the identity symbol $=$, an infinite list of individual variables $x_1, x_2, \ldots$, an infinite list of individual constants $a_1, a_2, \ldots$, and, for each positive integer n, an infinite list of n-ary predicate symbols $P_1^n, P_2^n, \ldots$, and the technical signs: (,). *Terms* and *declarative well-formed formulae* (d-wffs for short) of $\mathscr{L}$ are defined as usual.

We shall use the letters A, B, C (with subscripts if needed) as metalinguistic variables for both d-wffs of $\mathscr{L}$ and d-wffs of the extensions of $\mathscr{L}$ presented below. A d-wff with no free variables is said to be a *sentence*; otherwise a d-wff is said to be a *sentential function.* A (metalinguistic) expression of the form $A(x_{i_1}, \ldots, x_{i_n})$ refers to the sentential functions whose free variables are exactly the (explicitly listed) variables $x_{i_1}, \ldots, x_{i_n}$.

In order to obtain formal languages which are applicable in the analysis of the questions we are interested in we have to enrich the language $\mathscr{L}$. We shall do it in two steps. First, we will extend the language $\mathscr{L}$ to a language $\mathscr{L}_1$ in which we can express the so-called numerical questions; this category includes most of the "which" questions. Second, we will enrich the language $\mathscr{L}$ to a language $\mathscr{L}_2$ in which some propositional questions can be expressed.

1.3. Simple numerical questions. In order to illustrate the basic idea which underlies Kubiński's analysis of simple numerical questions let us take a look at the following table (k stands here for a positive integer, whereas x is an individual variable):

Table 1

(1.a) *For some* x, $A(x)$.	(1.b) *For which [at least one]* x, $A(x)$?
	(1.c) *For which [all]* x, $A(x)$?
(2.a) *For at least* kx, $A(x)$.	(2.b) *For which [at least* k*]* x, $A(x)$?
	(2.c) *Which are all [at least* k*]* x *such that* $A(x)$?
(3.a) *For more than* kx, $A(x)$.	(3.b) *For which [more than* k*]* x, $A(x)$?
	(3.c) *Which are all [more than* k*]* x *such that* $A(x)$?
(4.a) *For exactly* kx, $A(x)$.	(4.b) *For which [exactly* k*]* x, $A(x)$?
	(4.c) *Which are all [exactly* k*]* x *such that* $A(x)$?

The expressions which occur in the right column of the above table represent the paraphrases of *simple numerical questions*; each such question consists of a sentential function $A(x)$ and a *simple numerical operator*. The left column shows, however, that there is an analogy between the structure of simple numerical questions and the structure of some first-order (numerical) sentences. Yet, each numerical declarative sentence is paired with two simple numerical questions.

Let us now be more formal. First, let us extend the vocabulary of the initial language $\mathscr{L}$ in order to obtain the vocabulary of a certain new language $\mathscr{L}_1$. The extension goes on by adding the following symbols: an infinite list of numerals 1, 2, 3, ..., the constants $\leqslant$, $<$, C, and the technical sign / (slash). The declarative well-formed formulae (d-wffs) of $\mathscr{L}_1$ are precisely the d-wffs of the (initial) language $\mathscr{L}$. The *interrogative operators* of the language $\mathscr{L}_1$ (also called *simple numerical operators*) are expressions of the following forms:

$k \leqslant x_i$	(*for which* [*at least k*] x_i),
$k < x_i$	(*for which* [*more than k*] x_i),
kx_i	(*for which* [*exactly k*] x_i),
Cx_i	(*for which* [*all*] x_i),
$(k \leqslant)x_i$	(*which are all* [*at least k*] x_i *such that*),
$(k <)x_i$	(*which are all* [*more than k*] x_i *such that*),
$(k)x_i$	(*which are all* [*exactly k*] x_i *such that*).

where x_i is an individual variable and $k = 1, 2, \ldots$.

A *simple numerical question* is an expression of the form:

(1.3.0) $O(x_i)A(x_i)$,

where $A(x_i)$ is a sentential function with x_i as the only free variable and $O(x_i)$ is a simple numerical operator containing x_i as the only variable.

Let us now consider some examples.

The questions:

(1.3.1) Which Polish towns, at least four, are larger than Poznań?

(1.3.2) What are at least four Polish towns that are larger than Poznań?

may be regarded as falling under the scheme:

(1.3.3) $4 \leqslant x_i A(x_i)$.

The question:

(1.3.4) Which Polish towns, more than three, are larger than Poznań?

may be regarded as having the logical form of:

(1.3.5) $3 < x_i A(x_i)$.

The questions:

(1.3.6) Which four Polish towns are larger than Poznań?

(1.3.7) Exactly which four Polish towns are larger than Poznań?

fall under the scheme:

(1.3.8) $4x_i\, A(x_i)$.

The questions (1.3.1), (1.3.2), (1.3.4), (1.3.6) and (1.3.7) include some numerical expressions. On the other hand, questions with such expressions are not frequent in an ordinary discourse. Yet, many questions of an ordinary discourse have numerical reference components expressed by the grammar. Let us consider:

(1.3.9) Which Polish town is larger than Poznań?

(1.3.10) Which Polish towns are larger than Poznań?

By and large, the question (1.3.9) calls for exactly one example of a Polish town, whereas the question (1.3.10) calls for at least two examples of Polish towns. Thus the formalization of (1.3.9) should be of the form:

(1.3.11) $1x_i\, A(x_i)$,

whereas the formalization of (1.3.10) is either of the form:

(1.3.12) $2 \leqslant x_i\, A(x_i)$

or of the form:

(1.3.13) $1 < x_i\, A(x_i)$.

Besides the numerical components, many natural language questions include also completeness-claim components. Let us consider the following questions:

(1.3.14) Which are all of the Polish towns larger than Poznań?

(1.3.15) Which are all of the at least four Polish towns larger than Poznań?

(1.3.16) Which are all of the more than three Polish towns larger than Poznań?

(1.3.17) Which are all of the exactly four Polish towns larger than Poznań?

The formalizations of (1.3.14) - (1.3.17) are respectively of the form:

(1.3.18) $Cx_i\, A(x_i)$,

(1.3.19) $(4 \leqslant)\, x_i\, A(x_i)$,

(1.3.20) $(3 <)\, x_i\, A(x_i)$,

(1.3.21) $(4)\, x_i\, A(x_i)$.

The question:

(1.3.22) Which is the unique Polish town larger than Poznań?

has, in turn, a formal counterpart of the form:

(1.3.23) $(1)\, x_i\, A(x_i)$.

Kubiński stresses that some questions which do not express explicitly the completeness requirement can nevertheless be understood as complete-list questions. So it is also possible to conceive the questions (1.3.2) and (1.3.4) as falling under the schemata (1.3.19) and (1.3.20), respectively, and the questions (1.3.6) and (1.3.7) as having the form of (1.3.21). Similarly, the question (1.3.9) may correspond either to (1.3.11) or to (1.3.23).

Kubiński assigns to each simple numerical question a *set of direct answers*. This assignment is made in purely syntactical terms; a direct answer can be obtained from the question by means of simple syntactical transformations. At the same time, however, direct answers are "these sentences which everybody who understands the question ought to be able to recognize as *the simplest, most natural, admissible answer to this question*" ([25], p. 12; emphasis added). Let us designate by $A(x_i/a_j)$ the sentence which results from the sentential function $A(x_i)$ by proper substitution of the individual constant a_j for the (each free occurrence of) variable x_i. Direct answers to simple numerical questions can be characterized by means of the following table (in each case considered below $a_{j_1}, \ldots, a_{j_k}, \ldots, a_{j_n}$ are assumed to be nonequiform, i.e. syntactically different individual constants; x_z represents a variable which is substitutable for x_i in $A(x_i)$ and $A(x_z)$ is the result of the substitution):

Table 2

Question	*Direct answers*
$k \leqslant x_i\, A(x_i)$	$A(x_i/a_{j_1})$ & … & $A(x_i/a_{j_n})$, where $n \geqslant k$.
$k < x_i\, A(x_i)$	$A(x_i/a_{j_1})$ & … & $A(x_i/a_{j_n})$, where $n > k$.
$kx_i\, A(x_i)$	$A(x_i/a_{j_1})$ & … & $A(x_i/a_{j_k})$.
$Cx_i\, A(x_i)$	$A(x_i/a_{j_1})$ & … & $A(x_i/a_{j_n})$ & $\forall x_z(A(x_z) \rightarrow x_z = a_{j_1} \vee \ldots \vee x_z = a_{j_n})$.
$(k \leqslant)\, x_i\, A(x_i)$	$A(x_i/a_{j_1})$ & … & $A(x_i/a_{j_n})$ & $\forall x_z(A(x_z) \rightarrow x_z = a_{j_1} \vee \ldots \vee x_z = a_{j_n})$, where $n \geqslant k$.
$(k <)\, x_i\, A(x_i)$	$A(x_i/a_{j_1})$ & … & $A(x_i/a_{j_n})$ & $\forall x_z(A(x_z) \rightarrow x_z = a_{j_1} \vee \ldots \vee x_z = a_{j_n})$, where $n > k$.
$(k)\, x_i\, A(x_i)$	$A(x_i/a_{j_1})$ & … & $A(x_i/a_{j_k})$ & $\forall x_z(A(x_z) \rightarrow x_z = a_{j_1} \vee \ldots \vee x_z = a_{j_k})$.

In extensional semantics different individual constants may refer to the same object. Yet, Kubiński does not include the appropriate distinctness-claims (i.e. the formulas saying that different individual constants that occur in a given direct answer refer to different elements of the domain) into direct answers. The distinctness-claims, however, are elements of the so-called *inequality counterparts of direct answers*. Their definitions are straightforward.

In the paper [24] the analysis of simple numerical questions is extended and questions with the interrogative operators "for which [at most k] x_i", "for which all [at most k] x_i", "for which [at least k, but at most m] x_i" and "for which all [at least k, but at most m] x_i" (where $m > k$) are entered into the picture. The analysis of these questions goes on through the lines presented above; the definitions of direct answers are straightforward.

1.4. Propositional questions. The naturalness of the "interrogative operator-sentential function" analysis of numerical questions is due to the structural similarity of these questions and some quantified sentences. The situation is different, however, in the case of most propositional questions: one may argue that these questions do not contain expressions which may be analyzed as "binding" any variables. Yet, Kubiński shows that also these questions can be analyzed within the "interrogative operator-sentential function" framework: the leading idea of his analysis is that in the case of propositional questions the appropriate variables are sentential connective variables, that is, variables ranging over some (extensional) sentential connectives. But there is a price: the language of analysis must be enriched with some extensional connectives which are seldom used in logical texts (although these connectives are definable in first-order logic) as well as with some corresponding variables. We shall label this language by $\mathcal{L}_2$.

The vocabulary of the language $\mathcal{L}_2$ consists of all the symbols of the (basic) language $\mathcal{L}$ plus the following connectives: $\boldsymbol{as}$, $\boldsymbol{i}$, $\boldsymbol{k}$, $\boldsymbol{n}$, $\boldsymbol{b}_i^n$, $\boldsymbol{d}_{i_1,\ldots,i_k}^n$ (where $i \leqslant n$; $i_1 < \ldots < i_k$; $i_k \leqslant n$ and the sequence $i_1, \ldots, i_k$ can be empty). In addition, the vocabulary of $\mathcal{L}_2$ contains the variables: α, ε, η, β^n, δ^n (where $n \geqslant 1$), and the square brackets [ ]. The d-wffs (i.e. declarative well-formed formulae) of $\mathcal{L}_2$ can be defined as follows:

(a) each d-wff of $\mathcal{L}$ is a d-wff of $\mathcal{L}_2$;

(b) if A, B are d-wffs of $\mathcal{L}_2$, then $\neg A$, $\forall x_i A$, $\exists x_i A$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \equiv B)$, $\boldsymbol{as}A$, $\boldsymbol{i}AB$, $\boldsymbol{k}AB$, $\boldsymbol{n}AB$, αA, ηAB, εAB are d-wffs of $\mathcal{L}_2$;

(c) if $A_1, \ldots, A_n$ are d-wffs of $\mathcal{L}_2$, then $\boldsymbol{b}_i^n A_1, \ldots, A_n$, $\boldsymbol{d}_{i_1,\ldots,i_k}^n A_1, \ldots, A_n$, $\beta^n A_1, \ldots, A_n$, $\delta^n A_1, \ldots, A_n$ are d-wffs of $\mathcal{L}_2$;

(d) there are no other d-wffs of $\mathcal{L}_2$.

The connectives $\boldsymbol{as}$, $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{n}$, $\boldsymbol{b}_i^n$, $\boldsymbol{d}_{i_1,\ldots,i_k}^n$ are understood according to the following definitions:

DF. 1. $\boldsymbol{as}A \equiv A$.

DF. 2. $\boldsymbol{k}AB \equiv A \;\&\; B$.

DF. 3. $\boldsymbol{i}AB \equiv A \;\&\; \neg B$.

DF. 4. $\boldsymbol{n}AB \equiv \neg A$.

DF. 5. $\boldsymbol{b}_i^n A_1, \ldots, A_n \equiv \boldsymbol{as}A_i$.

DF. 6. $\boldsymbol{d}_{i_1,\ldots,i_k}^n A_1, \ldots, A_n \equiv B_1 \;\&\; \ldots \;\&\; B_n$, where B_r $(1 \leqslant r \leqslant n)$ is equal to

$\boldsymbol{as}A_r$ if r is one of the indices $i_1, \ldots, i_k$; if r is not one of the indices $i_1, \ldots, i_k$ or the sequence $i_1, \ldots, i_k$ is empty, then B_r is equal to $\neg A_r$.

The variable α ranges over the set $\{\boldsymbol{as}, \neg\}$, whereas the variables ε and η range over the sets $\{\boldsymbol{n}, \boldsymbol{i}, \boldsymbol{k}\}$ and $\{\boldsymbol{i}, \boldsymbol{k}\}$, respectively. A variable of the form β^n refers to connectives of the form $\boldsymbol{b}_i^n$. A variable δ^n ranges over the connectives of the form $\boldsymbol{d}^n_{i_1,\ldots,i_k}$.

Questions of the language $\mathscr{L}_2$ have the following forms:

(1.4.1) $[\alpha]\ \alpha A$,

(1.4.2) $[\beta^n]\ \beta^n A_1, \ldots, A_n$,

(1.4.3) $[\delta^n]\ \delta^n A_1, \ldots, A_n$,

(1.4.4) $[\varepsilon]\ \varepsilon AB$,

(1.4.5) $[\eta]\ \eta AB$,

where $A, B, A_1, \ldots, A_n$ are sentences (i.e. d-wffs without free variables) of $\mathscr{L}_2$.

We shall use the term *propositional questions* for the questions of $\mathscr{L}_2$. There are no standard readings of the interrogative operators that occur in propositional questions: these operators may represent various question-forming expressions. Yet, using the logical jargon propositional questions can be read as follows:

Table 3

Question	*Possible reading*
$[\alpha]\ \alpha A$	*Is it the case that A?*
$[\beta^n]\ \beta^n A_1, \ldots, A_n$	*Is it the case that A_1, or is it the case that $A_2, \ldots,$ or is it the case that A_n?*
$[\delta^n]\ \delta^n A_1, \ldots, A_n$	*Is it the case that A_1, and $A_2, \ldots,$ and A_n?*
$[\varepsilon]\ \varepsilon AB$	*Is it the case that A?; if so, is it also the case that B?*
$[\eta]\ \eta AB$	*It is the case that A; is it also the case that B?*

Each propositional question is accompanied with a set of direct answers to it. Again, direct answers are sentences of $\mathscr{L}_2$ which can be obtained from the question by some simple syntactical transformations; at the same time they are assumed to be the simplest possible answers. The following table characterizes direct answers to propositional questions:

Table 4

Question	*Direct answers*
$[\alpha]\ \alpha A$	$\boldsymbol{as}A,\ \neg A$
$[\beta^n]\ \beta^n A_1, \ldots, A_n$	$\boldsymbol{b}_i^n A_1, \ldots, A_n$, where $i = 1, 2, \ldots, n$.

$[\delta^n]$ $\delta^n A_1, \ldots, A_n$	$\boldsymbol{d}^n_{i_1,\ldots,i_k} A_1, \ldots, A_n$
$[\varepsilon]$ εAB	$\boldsymbol{n}AB$, $\boldsymbol{k}AB$, $\boldsymbol{i}AB$
$[\eta]$ ηAB	$\boldsymbol{k}AB$, $\boldsymbol{i}AB$

Questions of the form $[\alpha]$ αA may be regarded as *simple yes-no questions* (let us recall that $\boldsymbol{as}A$ is equivalent to A). Since $\beta^n_i A_1, \ldots, A_n$ is equivalent to $\boldsymbol{as}A_i$ and hence to A_i, questions of the form $[\beta^n]$ $\beta^n A_1, \ldots, A_n$ may be called *disjunctive questions.* A sentence of the form $\boldsymbol{b}^n_{i_1,\ldots,i_k} A_1, \ldots, A_n$ is equivalent to the conjunction B_1 & ... & B_n, where B_r $(1 \leqslant r \leqslant n)$ is either of the form $\boldsymbol{as}A_r$ or of the form $\neg A_r$ (to be more precise, B_r is equal to $\boldsymbol{as}A_r$ if r is one of the indices $i_1, \ldots, i_k$, and equal to $\neg A_r$ otherwise; if the sequence $i_1, \ldots, i_k$ is empty, then B_1 & ... & B_n is $\neg A_1$ & ... & $\neg A_n$). Thus, roughly, questions having the form $[\delta^n]$ $\delta^n A_1, \ldots, A_n$ ask about the logical value of each of the sentences $A_1, \ldots, A_n$. We shall call them *conjunctive questions.* Questions of the form $[\varepsilon]$ εAB and $[\eta]$ ηAB are called *conditional questions.* The direct answers to a question $[\varepsilon]$ εAB are the sentences $\boldsymbol{n}AB$, $\boldsymbol{k}AB$ and $\boldsymbol{i}AB$ which, in turn, are equivalent to the sentences $\neg A$, A & B, and A & $\neg B$, respectively; hence Kubiński calls them *conditional questions with revocable antecedents.* The direct answers to questions of the form $[\eta]$ ηAB are the sentences $\boldsymbol{k}AB$ and $\boldsymbol{i}AB$, which are equivalent to A & B and A & $\neg B$, respectively; thus these questions are called *conditional questions with irrevocable antecedents.*

Some examples may clarify matters here. The formal counterpart of the question:

(1.4.6) Is John handsome?

is of the form $[\alpha]$ αA. The question:

(1.4.7) Is John handsome or intelligent?

falls under the scheme $[\beta^2]$ $\beta^2 A, B$. But the question:

(1.4.8) Is John handsome and intelligent?

has a formal counterpart of the form $[\delta^2]$ $\delta^2 A, B$. The question:

(1.4.9) If Mary is John's mother, then is he the son of Peter?

has two possible readings. Under the first reading the question (1.4.9) can be answered by the sentence "Mary isn't John's mother"; thus the formal counterpart of (1.4.9) is of the form $[\varepsilon]$ εAB. The second reading does not allow for answering (1.4.9) with the negation of the "antecedent"; in this case the analyzed question falls under the scheme $[\eta]$ ηAB.

The analysis of propositional questions is furthered in [24]. This paper, however, simplifies and generalizes the concept of propositional question: the interrogative operators contain sentential connectives instead of sentential connective variables. It is assumed that the language of analysis is a first-order language which contains all the extensional connectives definable in the first-order logic; moreover, it is assumed that some linear ordering in each set of sentential connectives with the same number of arguments has been established. A propositional question is then defined as an expression of the form:

$$(*) \qquad (f_1, \ldots, f_n)\, A_1 \ldots A_k,$$

where $f_1, \ldots, f_n$ are k-argument sentential connectives, f_i precedes f_{i+1} with respect to the established linear ordering and $A_1, \ldots, A_k$ are sentences of the language of analysis. A *direct answer* to a question of the form $(*)$ is a sentence having the form:

$$(**) \qquad f_i A_1 \ldots A_k,$$

where $i \leqslant n$.

1.5. Compound numerical questions. It is not the case that each "which" question corresponds to some simple numerical question. It often happens that we ask not about a number of individuals having a certain property, but about couples, triples etc. of individuals bearing some relation. Moreover, some of such questions include also a (explicit or implicit) completeness-claim. But the situation is even more complicated, since many multiple "which" questions contain some numerical components which pertain to some of their constituents, but not to the whole question. Furthermore, many multiple "which" questions are ambiguous. Let us consider the question:

(1.5.1) Which three boys love which two girls?

The question (1.5.1) may be interpreted as asking about three boys and two girls such that each of these girls is loved by exactly one of the boys. But it is also possible to understand (1.5.1) as asking about three boys and two girls such that each of the boys loves the two girls. Moreover, it is as well possible to construe (1.5.1) as asking about three boys and some (but at least two and no more than six) girls such that each of the boys loves two of the girls. Sometimes (1.5.1) can also be understood as asking about a complete list of boys and girls fulfilling one of the above conditions.

One may argue that the ambiguity of (1.5.1) is due to the numerical expressions which occur in it. But let us consider the question:

(1.5.2) Which boys love which girls?

Again, by putting (1.5.2) a questioner may require information about (1) girl(s) loved by each of the boys, or (2) boys and girls such that each of the girls is loved by each of the boys. There are also further possible readings of (1.5.2).

The question (1.5.2) may be regarded is a special case of:

(1.5.3) Which [at least k] boy(s) love(s) which [at least n] girl(s)?

Similarly, the question (1.5.1) falls under the scheme:

(1.5.4) Which k boy(s) love(s) which n girl(s)?

Questions of the form (1.5.3) and (1.5.4) are ambiguous; the same holds true for most of the multiple "which" questions that contain numerical expressions and for many of them that do not contain such expressions.

Kubiński deals with the problems raised above by defining *compound interrogative operators*. These operators consist of strings of the following expressions: $k \leqslant$ ("for which [at least k]"), $k <$ ("for which [more than k]"), k ("for which [exactly k]"), C ("for which [all]"), $(k \leqslant)$ ("which are all [at least k] ... such that"), $(k <)$ ("which are all [more than k] ... such that"), (k) (which are all [exactly k] ... such that"), where $k \geqslant 1$, as well as of individual variables, some additional constants and the technical sign / (slash). The language of analysis of compound numerical questions is again the language $\mathscr{L}_1$ (cf. Section 1.3). For the sake of simplicity we shall restrict ourselves to the presentation of Kubiński's analysis of two-argument compound numerical questions.

A *two-argument compound numerical question* is an expression of $\mathscr{L}_1$ having the following form:

(1.5.5) $\quad {}^{\lambda}\Theta/\Xi\, x_i x_j\, A(x_i, x_j)$,

where $\lambda = 1, 2, 3$, $i \neq j$, $A(x_i, x_j)$ is a sentential function of $\mathscr{L}_1$ with x_i, x_j as the only free variables, and both Θ and Ξ may be of one of the following forms: $k \leqslant$, $k <$, k, C, $(k \leqslant)$, $(k <)$, (k), where $k = 1, 2, \ldots$ Thus each two-argument compound numerical question consists of a *two-argument compound interrogative operator* ${}^{\lambda}\Theta/\Xi\, x_i x_j$ and a sentential function whose only free variables are the variables of the corresponding interrogative operator. Since the natural-language numerical "which" questions are ambiguous, there are no standard readings of the two-argument interrogative operators; yet, by defining the set of direct answers to a given two-argument compound numerical question we can clarify its meaning. Again, direct answers are defined in syntactic terms, but they are regarded as the simplest, most natural admissible answers.

In order to go on we need some notational conventions. An expression of the form:

(*) $$\prod_{i=1}^{n} A_i$$

is an abbreviation of the conjunction:

(**) $$A_1 \,\&\, \ldots \,\&\, A_n,$$

whereas an expression of the form:

(***) $$\sum_{i=1}^{n} A_i$$

is an abbreviation of the disjunction:

(****) $$A_1 \vee \ldots \vee A_n.$$

Of course if $n = 1$, then both (*) and (***) reduce to A_1. By $A(x_i/a_s, x_j/a_t)$ we shall designate the sentence which results from the sentential function $A(x_i, x_j)$ by proper substitution of a_s for x_i and a_t for x_j, respectively.

Let us now analyze some two-argument compound numerical questions in detail. A question of the form:

(1.5.6) $\quad {}^1m/n\, x_i\, x_j\, A(x_i, x_j)$

has as direct answers sentences of the form:

(1.5.7) $$\prod_{k=1}^{r} A(x_i/a_{i_1}, x_j/a_{\alpha_k}) \;\&\; \prod_{k=1}^{s} A(x_i/a_{i_2}, x_j/a_{\beta_k}) \;\&\; \ldots \;\&\; \prod_{k=1}^{t} A(x_i/a_{i_m}, x_j/a_{\gamma_k}),$$

where each of the following sequences of individual constants: $(c)\; a_{i_1}, \ldots, a_{i_m}$, $(c_1)\; a_{\alpha_1}, \ldots, a_{\alpha_r}$, $(c_2)\; a_{\beta_1}, \ldots, a_{\beta_s}, \ldots, (c_h)\; a_{\gamma_1}, \ldots, a_{\gamma_t}$ has no repetitions and the sequence $(c*)\; a_{\alpha_1}, \ldots, a_{\alpha_r}, a_{\beta_1}, \ldots, a_{\beta_s}, \ldots, a_{\gamma_1}, \ldots, a_{\gamma_t}$ has exactly n different terms. Thus, roughly, a direct answer to a question of the form (1.5.6) specifies m different values of x_i and assigns to each of these values some value(s) of x_j in such a way that the total number of the assigned different values of x_j in equal to n. This is exactly the situation mentioned in the case of the first possible reading of the question (1.5.1). (Of course, different individual constants may designate the same object, so the description given above is only partly adequate. Yet, we shall disregard this property of extensional semantics in our comments.)

The direct answers to a question of the form:

(1.5.8) $\quad {}^2m/n\, x_i\, x_j\, A(x_i, x_j)$

are of the form:

(1.5.9) $$\prod_{r=1}^{n} \prod_{k=1}^{m} A(x_i/a_{i_k}, x_j/a_{j_r}),$$

where all the individual constants $a_{i_1}, \ldots, a_{i_m}$ are different and all the individual constants $a_{j_1}, \ldots, a_{j_n}$ are different. Thus, roughly, a direct answer to a question of the form (1.5.8) specifies m different values of x_i and assigns to each of these values the same collection of exactly n different values of x_j. This corresponds, among other things, to the second reading of the question (1.5.1).

A question of the form:

(1.5.10) $\quad {}^3m/n\, x_i\, x_j\, A(x_i, x_j)$

has as direct answers sentences of the form:

(1.5.11) $$\prod_{k=1}^{n} A(x_i/a_{i_1}, x_j/a_{\alpha_k}) \;\&\; \prod_{k=1}^{n} A(x_i/a_{i_2}, x_j/a_{\beta_k}) \;\&\; \ldots \;\&\; \prod_{k=1}^{n} A(x_i/a_{i_m}, x_j/a_{\gamma_k})$$

where each of the following sequences of individual constants: (c) $a_{i_1}, \ldots, a_{i_m}$, (c_1) $a_{\alpha_1}, \ldots, a_{\alpha_n}$, (c_2) $a_{\beta_1}, \ldots, a_{\beta_n}, \ldots$ (c_h) $a_{\gamma_1}, \ldots, a_{\gamma_n}$ has no repetitions. Thus a direct answer to a question of the form (1.5.10) specifies exactly m different values of x_i and assigns to each of these values exactly n different values of x_j; it is neither assumed nor denied that the relevant values of x_j are the same in all cases.

Let us now analyze questions of the form:

(1.5.12) $\quad {}^1C/C\ x_i x_j\ A(x_i, x_j)$,

(1.5.13) $\quad {}^2C/C\ x_i x_j\ A(x_i, x_j)$,

(1.5.14) $\quad {}^3C/C\ x_i x_j\ A(x_i, x_j)$.

Direct answers to (1.5.12) have the form of:

(1.5.15)
$$A(x_i/a_{i_1}, x_j/a_{j_1}) \mathbin{\&} \ldots \mathbin{\&} A(x_i/a_{i_n}, x_j/a_{j_m}) \mathbin{\&} \forall x_t \forall x_s \big(A(x_t, x_s) \rightarrow \big((x_t = a_{i_1}) \mathbin{\&} (x_s = a_{j_1})\big) \vee \ldots \vee \big((x_t = a_{i_n}) \mathbin{\&} (x_s = a_{j_m})\big)\big),$$

where $n \geqslant 1$ and the sentences $A(x_i/a_{i_1}, x_j/a_{j_1}), \ldots, A(x_i/a_{i_n}, x_j/a_{j_m})$ are syntactically different. Thus we may say that a direct answer to a question of the form (1.5.12) lists all the relevant values of x_i and x_j.

Direct answers to questions of the form (1.5.13) are sentences of the form:

(1.5.16)
$$\prod_{r=1}^{n} \prod_{k=1}^{m} A(x_i/a_{i_k}, x_j/a_{j_r}) \mathbin{\&} \prod_{r=1}^{n} \forall x_z \Big(A(x_z, x_j/a_{j_r}) \rightarrow \sum_{k=1}^{m} (x_z = a_{i_k})\Big) \mathbin{\&} \prod_{k=1}^{m} \forall x_e \Big(A(x_i/a_{i_k}, x_e) \rightarrow \sum_{r=1}^{n} (x_e = a_{j_r})\Big),$$

where $m, n \geqslant 1$, all the individual constants $a_{i_1}, \ldots, a_{i_m}$ are different and all the individual constants $a_{j_1}, \ldots, a_{j_n}$ are different. To speak generally, a direct answer to a question of the form (1.5.13) specifies all the values of x_i and assigns to each of these values the same collection of the relevant values of x_j.

Direct answers to questions of the form (1.5.14) fall under the following scheme:

(1.5.17)
$$\prod_{k=1}^{r} A(x_i/a_{i_1}, x_j/a_{\alpha_k}) \mathbin{\&} \forall x_z \Big(A(x_i/a_{i_1}, x_z) \rightarrow \sum_{k=1}^{r} (x_z = a_{\alpha_k})\Big)$$
$$\mathbin{\&} \prod_{k=1}^{s} A(x_i/a_{i_2}, x_j/a_{\beta_k}) \mathbin{\&} \forall x_z \Big(A(x_i/a_{i_2}, x_z) \rightarrow \sum_{k=1}^{s} (x_z = a_{\beta_k})\Big) \mathbin{\&} \ldots$$
$$\mathbin{\&} \prod_{k=1}^{t} A(x_i/a_{i_n}, x_j/a_{\gamma_k}) \mathbin{\&} \forall x_z \Big(A(x_i/a_{i_n}, x_z) \rightarrow \sum_{k=1}^{t} (x_z = a_{\gamma_k})\Big)$$
$$\mathbin{\&} \forall x_h \forall x_c \Big(A(x_h, x_c) \rightarrow \sum_{e=1}^{n} (x_h = a_{i_e})\Big),$$

where $n, r, s, \ldots, t \geqslant 1$ and each of the following sequences of individual constants: (c) $a_{i_1}, \ldots, a_{i_n}$, (c_1) $a_{\alpha_1}, \ldots, a_{\alpha_r}$, (c_2) $a_{\beta_1}, \ldots, a_{\beta_s}, \ldots, (c_f)$ $a_{\gamma_1}, \ldots, a_{\gamma_t}$, has no repetitions. Thus, roughly, a direct answer to a question of the form (1.5.14) specifies all the relevant values of x_i and assigns to each of these values a complete list of the corresponding values of x_j.

We will not present here the schemata of direct answers to the remaining two-argument compound numerical questions as well as the schemata of direct answers to more than two-argument compound numerical questions analyzed by Kubiński. Let us only notice that some generalization of the concept of compound numerical question is to be found in [22]; this paper modifies a bit the approach of the monographs [21] and [25]. According to the new proposal, compound numerical interrogative operators consist of some constants and strings of *interrogative quantifiers*; these quantifiers, in turn, are analogues of the simple numerical operators (see Section 1.3)

The vocabulary of the language $\mathscr{L}_3$ of the new analysis of compound numerical questions is the vocabulary of the language $\mathscr{L}_1$ (cf. Section 1.3) enriched with the square brackets [ ] as new technical signs. Terms and d-wffs of $\mathscr{L}_3$ are the terms and d-wffs of $\mathscr{L}_1$ (and thus of $\mathscr{L}$).

A *compound numerical question of the first kind* is an expression of $\mathscr{L}_3$ of the following form:

(1.5.18) $\quad [P_1, \ldots, P_r]\, k_1 x_{i_1} \ldots k_{r+1} x_{i_{r+1}} A(x_{i_1}, \ldots, x_{i_{r+1}})$,

where $A(x_{i_1}, \ldots, x_{i_{r+1}})$ is a sentential function of the language $\mathscr{L}_3$ whose only free variables are the (listed) variables $x_{i_1}, \ldots, x_{i_{r+1}}$, and $k_1, \ldots, k_{r+1}$ is a sequence of natural numbers which are greater than zero. $P_1, \ldots, P_r$, in turn, is a sequence of natural numbers that fulfills the following conditions:

(a) if $k_s \geqslant k_{s+1}$, then $P_s \leqslant k_{s+1}$;

(b) if $k_s < k_{s+1}$, then $t \leqslant P_s \leqslant k_{s+1}$, where t is the least natural number such that $t \times k_s \geqslant k_{s+1}$ ($\times$ is the multiplication sign here).

The expressions $k_1 x_{i_1}, \ldots, k_{r+1} x_{i_{r+1}}$ are called *interrogative quantifiers of the first kind*, whereas the sequence $P_1, \ldots, P_r$ is called *the sequence of indices*. The expression $[P_1, \ldots, P_r]\, k_1 x_{i_1} \ldots k_{r+1} x_{i_{r+1}}$ is a *compound numerical interrogative operator of the first kind*. The rationale of the conditions (a) and (b) is to guarantee the existence of at least one direct answer to a compound numerical question of the first kind. Roughly, the semantic function of an index P_s is to indicate that exactly $P_s x_{i_{s+1}}$ should be assigned to each of the distinguished $k_s x_{i_s}$. Thus a question of the form:

(1.5.19) $\quad [1]\ 1x_i 1x_j A(x_i, x_j)$

can be read:

(1.5.20) $\quad$ Which one x_i and one x_j are such that $A(x_i, x_j)$?

But a question of the form:

(1.5.21) $[1]\ 3x_i\, 1x_j\, A(x_i, x_j)$

can be read:

(1.5.21) Which three x_i and one x_j are such that $A(x_i, x_j)$?

The question:

(1.5.22) $[2]\ 3x_i\, 3x_j\, A(x_i, x_j)$?

can, in turn, be read as:

(1.5.23) Which three x_i and three x_j are such that $A(x_i, x_j)$ and each of the x_i is associated with exactly two x_j?

Similarly, the question:

(1.5.24) $[2, 1]\ 4x_i\, 3x_j\, 2x_k\, A(x_i, x_j, x_k)$

may be read as follows:

(1.5.25) Which four x_i, three x_j, and two x_k are such that $A(x_i, x_j, x_k)$ and each x_i is associated with two x_j, whereas each x_j is associated with exactly one x_k?

Of course, a question of the form:

(1.5.26) $[1, 1, \ldots, 1]\ 1x_{i_1}, \ldots, 1x_{i_n}\, A(x_{i_1}, \ldots, x_{i_n})$

may be also read as:

(1.5.27) For which one x_{i_1}, one x_{i_2}, ..., and one x_{i_n}, $A(x_{i_1}, \ldots, x_{i_n})$?

The paper [22] does not contain a systematic analysis of compound numerical questions whose operators contain interrogative quantifiers different from "for which [exactly k] x_i". These questions are examined by Leszko in [29] and [30], where the analysis of compound numerical questions is furthered. Some generalization can also be found in [4]. It is easily seen that direct answers to various compound numerical questions are very complicated and it is not a trivial enterprise to assign to each such question a set of direct answers. Leszko (cf. [29] and [30]) solved this problem for many questions by using tools borrowed from graph theory and the theory of matrices. There are also some close connections between compound interrogative operators and algebraic structures; the papers of Kubiński [22] and Graczyńska [4] provide some interesting results in this field.

2. SYSTEMS OF THE LOGIC OF QUESTIONS

2.1. The logical basis. One may doubt whether the considerations presented above belong to the *logic* of questions; the term "logical theory of questions" seems more appropriate here. But what is *the logic of questions* about? The opinions here are divided. Kubiński's own proposal may be briefly expressed as follows: the logic of questions consists of systems whose theses describe either some basic relations between questions or different relations of answerhood between declarative sentences and questions. We shall present here two systems of this kind built by Kubiński: the system ***S*** and the system ***OR***. These systems are not the only systems of the logic of questions which can be found in the papers of Kubiński; yet, they seem to be the most interesting from the intuitive point of view. Let us add that some systems of the logic of questions built up in the "spirit" of Kubiński can also be found in the papers of his student Robert Leszko (cf. References).

In order to go on we need some supplementary concepts.

First, let us construct some new formalized language $\mathscr{L}^*$. This language can be obtained from the languages $\mathscr{L}_1$ and $\mathscr{L}_2$ described above; its vocabulary is the sum of the vocabularies of $\mathscr{L}_1$ and $\mathscr{L}_2$. The declarative well-formed formulas (d-wffs) of $\mathscr{L}^*$ are simply the d-wffs of $\mathscr{L}_1$ and of $\mathscr{L}_2$. The questions of $\mathscr{L}^*$, however, are the propositional questions of the language $\mathscr{L}_2$, the simple numerical questions of the language $\mathscr{L}_1$ and the two-argument compound numerical questions of the language $\mathscr{L}_1$, exclusively.

Let us now supplement the language $\mathscr{L}^*$ with a standard model-theoretic semantics. To be more precise, an *interpretation* of $\mathscr{L}^*$ is an ordered pair $\langle \mathscr{M}, f \rangle$, where $\mathscr{M}$ is a non-empty set (called *the universe*) and f is an *interpretation function* defined on the set of individual constants and predicate symbols of $\mathscr{L}^*$ in the usual way. If $\mathfrak{M} = \langle \mathscr{M}, f \rangle$ is an interpretation of $\mathscr{L}^*$, then each infinite sequence of the elements of $\mathfrak{M}$ is called a $\mathfrak{M}$*-valuation*. The concept of *value of a term under a* $\mathfrak{M}$*-valuation* is defined in the standard way. The concept of *satisfaction of a d-wff by a* $\mathfrak{M}$*-valuation* is defined in the standard way for atomic d-wffs and d-wffs containing the usual logical constants $\neg$, &, $\vee$, $\rightarrow$, $\equiv$, $\forall$, $\exists$, but also in such a way that the equivalencies of the definitions Df. 1 - Df. 6 (see Section 1.4) are satisfied by each $\mathfrak{M}$-valuation for each interpretation $\mathfrak{M}$. (We hope that the Reader will forgive us this rough description of the definition of satisfaction; there is no room for going into details.) By and large, the "unusual" logical constants of $\mathscr{L}^*$ are then understood according to the definitions Df. 1 - Df. 6 formulated above. A d-wff A *of* $\mathscr{L}^*$ is *true in an interpretation* $\mathfrak{M}$ *of* $\mathscr{L}^*$ if and only if A is satisfied by each $\mathfrak{M}$-valuation. A *tautology* is a d-wff which is true in each interpretation of $\mathscr{L}^*$. A d-wff A is said to be *satisfiable* if and only if there is at least one interpretation $\mathfrak{M}$ such that A is satisfied by some $\mathfrak{M}$-valuation; otherwise A is a *contradiction*. By a *contradictory set of d-wffs* we mean any set of d-wffs such that for each interpretation $\mathfrak{M}$ there is no $\mathfrak{M}$-valuation which satisfies all the d-wffs of this set. A set of d-wffs X *logically entails* a d-wffs A if and only if for each interpretation $\mathfrak{M}$ and each $\mathfrak{M}$-valuation $\boldsymbol{v}$, if all the d-wffs in X are satisfied by $\boldsymbol{v}$, then A is also satisfied by $\boldsymbol{v}$. For some reason which will become clear in a moment we also have to introduce some other concept of entailment, namely, the concept of *regular entailment*. We say that a set of d-wffs X *regularly entails* a d-wff A if and only if (1) X is not a contradictory set and X logically entails A, or (2) X is a contradictory set and A is a contradiction.

The relation of regular entailment is a consequence operation (in the sense of Tarski) in each so-called homogeneous family of sets of d-wffs of $\mathscr{L}^*$; a family ϕ of sets of d-wffs is said to be homogeneous if and only if (1) ϕ is empty, or (2) ϕ is non-empty and no element of ϕ is a contradictory set of d-wffs, or (3) ϕ is non-empty and all the element(s) of ϕ are contradictory sets of d-wffs. For a detailed description of the properties of regular entailment see the paper of Kubiński [10].

We say that a d-wff A is *equivalent* to a d-wff B if and only if A logically entails B and B logically entails A.

We shall use the letters Q, R (with subscripts if needed) as metalinguistic variables for questions. The set of direct answers to a question Q (to a question R) will be referred to as dQ (as dR).

2.2. The system *S*. At a first approximation we may say that the theses of the system $\boldsymbol{S}$ tell us which questions are equipollent, which questions are stronger than others, which questions are weaker than others, and which questions are completely or partially independent from others. The analyzed relations between questions are defined on the metatheoretical level in semantic terms.

We say that a question Q is *equipollent* to a question R if and only if there exists a bijection $i: dQ \mapsto dR$ such that for each $A \in dQ$, A is equivalent to $i(A)$ (i.e. to the corresponding element of dR).

A question Q is *weaker than* a question R if and only if Q is not equipollent to R, but there exists a surjection $i: dR \mapsto dQ$ such that for each $B \in dR$, B regularly entails $i(B)$.

A question Q is *stronger than* a question R if and only if R is weaker than Q, but Q is not weaker than R.

A question Q is *completely independent from* a question R if and only if no element of dQ is regularly entailed by any element of dR and no element of dR is regularly entailed by any element of dQ.

A question Q is *partially independent from* a question R if and only if Q is not completely independent from R, Q is not equipollent to R, and Q is neither stronger nor weaker than R.

Note that if the relation of being stronger was defined by means of the concept of logical entailment, we would obtain the following counterintuitive consequence: each question whose all direct answers are contradictions is stronger than any question.

Let us now describe the language of the system $\boldsymbol{S}$; the symbolism presented below differs in some details from the original symbolism adopted by Kubiński.

The vocabulary of the language $\mathscr{L}_S$ of the system $\boldsymbol{S}$ contains all the expressions of the language $\mathscr{L}^*$ which enable us to form questions of this language and the following symbols: **Eqp** ("is equipollent to"), **Wk** ("is weaker than"), **Str** ("is stronger than"), **Cind** ("is completely independent from"), **Pind** ("is partially independent from"). In addition, the vocabulary of $\mathscr{L}_S$ contains the technical symbols ( ) (parentheses) and , (comma).

The *well-formed formulas* of the language $\mathscr{L}_S$ are all the expressions of the following five forms: **Eqp**(Q, R), **Wk**(Q, R), **Str**(Q, R), **Cind**(Q, R), **Pind**(Q, R), where Q and R are questions of $\mathscr{L}^*$ (i.e. propositional questions, simple numerical questions or two-argument compound numerical questions).

Let us stress that the language $\mathscr{L}_S$ does not contain compound formulas.

The system $\boldsymbol{S}$ *is defined as the set of all the true formulas of the language* $\mathscr{L}_S$. The formulas of the form **Eqp**(Q, R), **Wk**(Q, R), **Str**(Q, R), **Cind**(Q, R), **Pind**(Q, R) are true if and only if Q is equipollent to R, Q is weaker than R, Q is stronger than R, Q is completely independent from R, and Q is partially independent from R, respectively. We shall be using the letter $\boldsymbol{S}$ as referring to the set of all the theses of the system $\boldsymbol{S}$.

The following are examples of (meta)theorems of the system $\boldsymbol{S}$:

T1. $\mathbf{Eqp}([\alpha]\alpha A,\ [\beta^2]\beta^2 A,\ \neg A)$

T2. $\mathbf{Eqp}([\alpha]\alpha A,\ [\delta^1]\delta^1 A)$

T3. $\mathbf{Eqp}([\alpha]\alpha A,\ [\alpha]\alpha \neg A)$

T4. $\mathbf{Eqp}([\varepsilon]\varepsilon AB,\ [\beta^3]\beta^3 \neg A,\ A \,\&\, B,\ A \,\&\, \neg B)$

T5. $\mathbf{Eqp}([\eta]\ \eta AB,\ [\beta^2]\ \beta^2 A \,\&\, B,\ A \,\&\, \neg B)$

T6. $\mathbf{Eqp}([\varepsilon]\ \varepsilon AB,\ [\varepsilon]\ \varepsilon A\ \neg B)$

T7. $\mathbf{Eqp}([\eta]\ \eta AB,\ [\eta]\ \eta A \neg B)$

T8. $\mathbf{Eqp}([\delta^n]\,\delta^n A_1, \ldots, A_n, [\beta^k]\,\beta^k d^n A_1, \ldots, A_n, d^n_1 A_1, \ldots, A_n, \ldots, d^n_{1,\ldots,n} A_1, \ldots, A_n)$, *where* $k = 2^n$.

Let us observe that theorems T1, T4, T5 and T8 yield that each propositional question is equipollent to some disjunctive question.

A sentential function $A(x_{i_1}, \ldots, x_{i_n})$ is said to be *normal* just in case $A(x_{i_1}, \ldots, x_{i_n})$ is neither a tautology nor a contradiction and no sentence which results from $A(x_{i_1}, \ldots, x_{i_n})$ by proper substitution of individual constants for individual variables is a tautology or a contradiction. We assume that all the sentential functions which occur in the questions considered below are normal.

T9. *If* $k+1 = m$, *then* $\mathbf{Eqp}(k < x_i\ A(x_i),\ m \leqslant x_i\ A(x_i))$.

Theorem T9 yields that simple numerical questions with the operators "for which [more than k] x_i" and "for which [at least $k+1$] x_i" are equipollent. Thus one of those categories of questions is superfluous.

T10. $\mathbf{Eqp}({}^i 1/m\ x_k x_z A(x_k, x_z),\ {}^j 1/m\ x_k x_z A(x_k, x_z))$, *where* $m \geqslant 1$ *and* $1 \leqslant i, j \leqslant 3$.

Theorem T10 yields that if $A(x_k, x_z)$ is normal, then questions having the forms ${}^1 1/\mathrm{m}\ x_k x_z A(x_i, x_z)$, ${}^2 1/\mathrm{m}\ x_k x_z A(x_i, x_z)$, ${}^3 1/\mathrm{m}\ x_k x_z A(x_i, x_z)$ are pairwise equipollent.

T11. $\mathbf{Eqp}({}^1 m/1\ x_k x_z A(x_k, x_z),\ {}^2 m/1\ x_k x_z A(x_k, x_z))$, *where* $m \geqslant 1$.

T12. *If* $m > k$, *then* $\mathbf{Wk}(kx_i A(x_i),\ mx_i A(x_i))$.

T13. *If* $m > k$, *then* $\mathbf{Wk}(kx_i A(x_i), m \leqslant x_i A(x_i))$.

T14. *If* $m > k$, *then* $\mathbf{Wk}(kx_i A(x_i), m < x_i A(x_i))$.

T15. *If* $m > k$, *then* $\mathbf{Wk}(k \leqslant x_i A(x_i), m \leqslant x_i A(x_i))$.

T16. *If* $m > k$, *then* $\mathbf{Wk}(k \leqslant x_i A(x_i), m < x_i A(x_i))$.

T17. *If* $k+1 = m$, *then* $\mathbf{Wk}(k < x_i A(x_i), m < x_i A(x_i))$.

T18. *If* $k+1 = m$, *then* $\mathbf{Wk}(k < x_i A(x_i), m \leqslant x_i A(x_i))$.

T19. *If* $k+1 = m$, *then* $\mathbf{Wk}(kx_i A(x_i), m < x_i A(x_i))$.

T20. $\mathbf{Wk}({}^i m/1\, x_k x_z A(x_k, x_z), {}^3 m/1\, x_k x_z A(x_k, x_z))$, *where* $m > 1$ *and* $1 \leqslant i \leqslant 2$.

T21. $\mathbf{Wk}({}^1 m/n\, x_k x_z A(x_k, x_z), {}^2 m/n\, x_k x_z A(x_k, x_z))$, *where* $m > 1$ *and* $n > 1$.

T22. $\mathbf{Wk}({}^i m/n\, x_k x_z A(x_k, x_z), {}^i s/t\, x_k x_z A(x_k, x_z))$, *where* $1 \leqslant i \leqslant 3$ *and either* $1 \leqslant m < s$ *and* $1 \leqslant n \leqslant t$ *or* $1 \leqslant m \leqslant s$ *and* $1 \leqslant n < t$.

It is easy to find examples of the (meta)theorems with the constants **Str**, **Cind** and **Part**; we leave it to the reader.

One my ask whether the usage of the word "system" is justified in the case of $\boldsymbol{S}$; a deductive system is usually defined as a set of formulas closed under some consequence operation. Yet, Kubiński shows that there is a consequence operation $\boldsymbol{Cn}_s$ such that $\boldsymbol{S}$ is closed under $\boldsymbol{Cn}_s$. Let us designate by $\boldsymbol{J}(\boldsymbol{S})$ the set of all well-formed formulas of the language $\mathscr{L}^*$. The function $\boldsymbol{Cn}_s$ is defined as follows:

$$(*) \quad \boldsymbol{Cn}_s(X) = \boldsymbol{S} \quad \text{if} \quad X \subset \boldsymbol{S}; \quad \text{otherwise} \quad \boldsymbol{Cn}_s(X) = \boldsymbol{J}(\boldsymbol{S}).$$

It can be shown that $\boldsymbol{Cn}_s$ is a consequence operation (in the sense of Tarski) in the set $2^{\boldsymbol{J}(\boldsymbol{S})}$. It is obvious that the system $\boldsymbol{S}$ is consistent; it can be shown that $\boldsymbol{S}$ is also complete with respect to $\boldsymbol{Cn}_s$ and $\boldsymbol{J}(\boldsymbol{S})$. Of course, $\boldsymbol{S}$ is undecidable.

Although $\boldsymbol{S}$ is a deductive system, it is not an axiomatic system. It would be an interesting enterprise to axiomatize $\boldsymbol{S}$; yet, Kubiński did not suggest any solution to this problem.

2.3. The system *OR*. Let us now describe the system ***OR*** of "logic of answers" built by Kubiński. Roughly, the theses of this system describe different categories of answers to the analyzed questions. As in the case of the system $\boldsymbol{S}$, we shall modify a bit the original symbolism of Kubiński; this can make the reading of formulas easier.

The vocabulary of the language $\mathscr{L}_{\boldsymbol{OR}}$ of the system ***OR*** includes the vocabulary of the language $\mathscr{L}^*$ (see Section 2.2) and the following expressions: **da** ("is a direct answer to"), **ida** ("is an indirect answer to"), **ada** ("is an almost direct answer to"), **pida** ("is a proper indirect answer to"), **pta** ("is a partial answer to"), **ar** ("is an answer to ... relative to"). The *well-formed formulas* of the language $\mathscr{L}_{\boldsymbol{OR}}$ are the expressions of the following forms: **da**(A, Q), **ida**(A, Q), **ada**(A, Q), **pida**(A, Q), **pta**(A, Q), **ar**(A, Q, B), where A and B are d-wffs of the language $\mathscr{L}^*$ and Q is a question of the language $\mathscr{L}^*$. *The system* ***OR*** *is defined as the set of all the true well-formed formulas of the language* $\mathscr{L}_{OR}$. The appropriate truth-conditions are at the same time semantic definitions of various categories of answers (of course with the exception of the truth-conditions for the formulas of the form **da** (A, Q); here the definitions of direct answers to various questions are taken for granted). These truth conditions are as follows:

An expression of the form **da** (A, Q) is true iff A is a direct answer to Q.

An expression of the form **ida** (A, Q) is true iff A is not a direct answer to Q, but A regularly entails some direct answer to Q.

Thus an *indirect answer* is a d-wff which regularly entails some direct answer, but is not a direct answer itself. Let us observe that if the concept of indirect answer was defined by means of logical entailment, we would obtain the following paradoxical consequence: each contradiction which is not a direct answer to a question is an indirect answer to this question.

An expression of the form **ada** (A, Q) is true iff the expression **ida** (A, Q) is true and A is equivalent to some direct answer to Q.

It follows that *an almost direct answer* is an indirect answer which is equivalent to some direct answer.

An expression of the form **pida** (A, Q) is true iff the expression **ida** (A, Q) is true, but A is not equivalent to any direct answer to Q.

Thus a *proper indirect answer* is an indirect answer which is not an almost direct answer.

An expression of the form **pta** (A, Q) is true iff neither the expression **da** (A, Q) nor the expression **ida** (A, Q) is true, but there exists a d-wff B such that (i) the following expressions are *not* true: **da** (B, Q), **ida** (B, Q), and (ii) at least one of the following expressions *is* true: **da** $(A \& B, Q)$, **ida** $(A \& B, Q)$.

It follows that a *partial answer* is a d-wff which is neither a direct answer nor an indirect answer, but which in conjunction with some d-wff which is also neither a direct answer nor an indirect answer forms some direct or indirect answer.

An expression of the form **ar** (A, Q, B) is true iff the expression **da** $(A \& B)$ is true or the expression **ida** $(A \& B)$ is true.

According to the above condition, a d-wff A is *an answer to* a question Q *relative to* a d-wff B just in case the conjunction $A \& B$ is either direct or indirect answer to Q.

It is easy to find examples of theses of the system ***OR***.

Let us add that the set of theses of the system ***OR*** is closed under some consequence operation; moreover, the system ***OR*** is consistent and complete (with respect to this consequence operation). Yet, it is still an open problem to build up the system ***OR*** in an axiomatic form. The same holds true with respect to the other systems of the logic of questions proposed by Kubiński.

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