Towards a Uniform Account of Proofs and Refutations

Andrzej Wiśniewski

Abstract A set of well-formed formulas (wffs) is holistically inconsistent iff it is inconsistent, but each wff in the set is consistent. We present a sequent calculus for holistically inconsistent sets of wffs of Classical Propositional Logic. Since valid, inconsistent, and contingent wffs correspond to different, yet strictly defined, holistically inconsistent sets, a proof of a sequent based on holistically inconsistent set of a given kind can be regarded, depending on the case, as a proof of a valid wff, a refutation of an inconsistent wff, and a refutation of a contingent wff, respectively.

Key words: proofs, refutations, holistic inconsistency, sequent calculi

1 Introduction

Consider a formal language supplemented with a bivalent semantics rich enough to define some concept of truth of a well-formed formula (henceforth: wff) in a model. The expression "model" is used here as a cover term; depending on the particular form of the language, models are valuations of some kind, relational structures, and so on. Usually, a formal language has many models of a given kind. When a non-empty class of models, $\mathcal{M}$, is fixed, the set of all wffs of the language splits, first, into two disjoint subsets: $\mathtt{Val}^{\mathcal{M}}$ and $\mathtt{NVal}^{\mathcal{M}}$. The set $\mathtt{Val}^{\mathcal{M}}$ comprises all the wffs which are *valid* w.r.t. the class of models $\mathcal{M}$, that is, which are true in each model from $\mathcal{M}$. The set $\mathtt{NVal}^{\mathcal{M}}$, in turn, comprises all the remaining wffs, that is, wffs which are not valid w.r.t. the class of models $\mathcal{M}$. However, the set $\mathtt{NVal}^{\mathcal{M}}$ is far from being homogenous. It includes *inconsistent* (also called *unsatisfiable*) wffs, that is, wffs which are not true

Andrzej Wiśniewski
Department of Logic and Cognitive Science
Faculty of Psychology and Cognitive Science
Adam Mickiewicz University, Poznań, Poland
e-mail: Andrzej.Wisniewski@amu.edu.pl

in any model from the class $\mathcal{M}$. But it also includes wffs which are consistent (or *satisfiable*) without being valid, i.e. wffs which are true in some model(s) belonging to the class $\mathcal{M}$, but are not true in other models from the class. Following a philosophical rather than a logical tradition, let us call these wffs *contingent*. To be more precise, when a class of models $\mathcal{M}$ is fixed, the set $\mathtt{NVal}^{\mathcal{M}}$ splits into the set $\mathtt{Inc}^{\mathcal{M}}$ of $\mathcal{M}$-*inconsistent* wffs (i.e. wffs which are not true in any model from $\mathcal{M}$) and the set $\mathtt{Ctg}^{\mathcal{M}}$ of $\mathcal{M}$-*contingent* wffs, that is, wffs which are neither $\mathcal{M}$-valid nor $\mathcal{M}$-inconsistent.

Looking from the proof-theoretic point of view, the main challenge for a logician is to build a calculus which makes provable all the valid (w.r.t. a given class of models) wffs and only them. Sometimes, as a by-product, a calculus gives an account of inconsistent wffs as well. Analytic tableaux are paradigmatic examples here. However, contingent wffs remain beyond the scope of interest. The (still rare) advocates of refutation methods see the goal differently: they aim at proof-theoretic accounts of non-validities (cf. Skura (2009), Skura (2011)). But the class of non-validities includes both inconsistent wffs and contingent wffs. This distinction seems to play no role in refutation calculi, however. Last but not least, logical calculi focussed on validities and these focussed on non-validities operate with diverse formal means.

The aim of this short note is to present a calculus which, on the one hand, differentiates between proofs of valid wffs, refutations of inconsistent wffs, and refutations of contingent wffs. On the other hand, the calculus offers a uniform proof-mechanism. This is achieved by the introduction of a kind of conceptual unifier, namely the notion of a holistically inconsistent *set* of wffs. The system "calculates" such sets or, more precisely, sequents based on them. Since valid, inconsistent, and contingent wffs correspond to different, yet strictly defined, holistically inconsistent sets, a proof of a sequent based on a set of a given kind can be regarded, depending on the case, as a proof or as a refutation of the corresponding wff.

2 The logical basis

We remain at the level of Classical Propositional Calculus (CPL for short). As for the language of (the analysed version of) CPL, we assume that the vocabulary comprises a countably infinite set of propositional variables, the connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$, and brackets. *Well-formed formulas* (henceforth: wffs) of the language are defined as usual. We use A, B, C, D, with subscripts when needed, as metalanguage variables for wffs, and X, Y, with or without subscripts or superscripts, as metalanguage variables for sets of wffs. The letters p, q, r, s, t are exemplary elements of the set of propositional variables of the language.

Let $\mathbf{1}$ stand for truth and $\mathbf{0}$ for falsity. A CPL-*valuation* is a function from the set of wffs to the set $\{\mathbf{1}, \mathbf{0}\}$, satisfying the following standard conditions: (a) $v(\neg A) = \mathbf{1}$ iff $v(A) = \mathbf{0}$; (b) $v(A \vee B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ or $v(B) = \mathbf{1}$; (c) $v(A \wedge B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ and $v(B) = \mathbf{1}$; (d) $v(A \rightarrow B) = \mathbf{1}$ iff $v(A) = \mathbf{0}$ or $v(B) = \mathbf{1}$, (e) $v(A \leftrightarrow B) = \mathbf{1}$ iff $v(A) = v(B)$.

For brevity, in what follows we will be omitting references to CPL. By wffs we will mean wffs of the language of CPL, and by valuations we will mean CPL-valuations.

Definition 1 (Consistency, inconsistency, validity, and contingence). A set of wffs X is consistent iff there exists a valuation v such that for each $A \in X$, $v(A) = \mathbf{1}$; otherwise X is inconsistent. A wff B is:

1. consistent iff the singleton set $\{B\}$ is consistent,
2. inconsistent iff the singleton set $\{B\}$ is inconsistent,
3. valid iff for each valuation v, $v(B) = \mathbf{1}$,
4. contingent iff B is neither inconsistent nor valid.

CPL-entailment, $\models$, is defined as follows:

Definition 2 (Entailment). $X \models A$ iff for each valuation v:

- if $v(B) = \mathbf{1}$ for every $B \in X$, then $v(A) = \mathbf{1}$.

The next definition introduces the crucial notion.

Definition 3 (Holistically inconsistent set; HI-set). A set of wffs X is holistically inconsistent iff X is inconsistent, but each wff in X is consistent.

Observe that each HI-set has at least two elements.

The following are true:

Corollary 1. *A wff C is contingent iff $\{C, \neg C\}$ is a* HI*-set.*

Proof. ($\Rightarrow$) If C is a contingent wff, then there are valuations v, v^* such that $v(C) = \mathbf{1}$ and $v^*(C) = \mathbf{0}$. So both C and $\neg C$ are consistent wffs. On the other hand, the set $\{C, \neg C\}$ is inconsistent. Therefore $\{C, \neg C\}$ is a HI-set.
($\Leftarrow$) If $\{C, \neg C\}$ is a HI-set, then both C and $\neg C$ are consistent wffs. Thus C is a contingent wff. □

Corollary 2. *A wff C is inconsistent iff $\{C \vee p, C \vee \neg p\}$ is a* HI*-set.*

Proof. ($\Rightarrow$) Assume that C is an inconsistent wff. Each of the wffs: $C \vee p$, $C \vee \neg p$, is consistent, however. On the other hand, the set $\{C \vee p, C \vee \neg p\}$ is inconsistent and hence is a HI-set.
($\Leftarrow$) If $\{C \vee p, C \vee \neg p\}$ is a HI-set, it is an inconsistent set and hence $\{C \vee p, C \vee \neg p\} \models p \wedge \neg p$. It follows that $\{C, C \vee \neg p\} \models p \wedge \neg p$ and therefore $C \models p \wedge \neg p$. Thus C is inconsistent. □

Corollary 3. *A wff C is valid iff $\{\neg C \vee p, \neg C \vee \neg p\}$ is a* HI*-set.*

Proof. ($\Rightarrow$) If C is valid, then $\neg C$ is inconsistent. But, similarly as before, both $\neg C \vee p$ and $\neg C \vee \neg p$ are consistent wffs, and the set $\{\neg C \vee p, \neg C \vee \neg p\}$ is inconsistent. Therefore $\{\neg C \vee p, \neg C \vee \neg p\}$ is a HI-set.
($\Leftarrow$) The set $\{\neg C \vee p, \neg C \vee \neg p\}$, as a HI-set, is inconsistent. Thus $\{\neg C \vee p, \neg C \vee \neg p\} \models p \wedge \neg p$ and therefore $\neg C \models p \wedge \neg p$. It follows that $\neg C$ is an inconsistent wff and hence C is a valid wff. □

Thus validity, inconsistency and contingency of wffs are expressible in terms of HI-sets.

3 The system $\mathsf{HI}^{\mathsf{CPL}}$

Since the system we are going to present "calculates" HI-sets, we label it by $\mathsf{HI}^{\mathsf{CPL}}$.

We operate with *sequents* of the form $Y \vdash$, where Y is an at least two-element finite set of CPL-wffs. In practice, we write down a sequent $Y \vdash$ by listing the elements of Y left to the turnstile. An inscription of the form '$C \in \mathsf{CPL}$' means 'C is a thesis of CPL', i.e. is provable in CPL.

A sequent, $Y \vdash$, is in the *normal form* iff each $C \in Y$ is in the disjunctive normal form (hereafter: DNF).

An *axiom* of $\mathsf{HI}^{\mathsf{CPL}}$ is a sequent $Y \vdash$ such that each $B \in Y$ is an elementary conjunction, a conjunction of all the wffs in Y involves complementary literals, and no $B \in Y$ involves complementary literals. Here are examples of axioms:

$$p, \neg p \vdash \tag{1}$$

$$\neg p \wedge \neg q, p \ \vdash \tag{2}$$

$$\neg p \wedge \neg q, q \ \vdash \tag{3}$$

$$\neg p \wedge \neg q, p \wedge \neg q, q \wedge \neg p \ \vdash \tag{4}$$

There are only two (primary) *rules* of $\mathsf{HI}^{\mathsf{CPL}}$, namely:

R_1: $\dfrac{Y \cup \{A\} \vdash \qquad Y \cup \{B\} \vdash}{Y \cup \{A \vee B\} \vdash}$

R_2: $\dfrac{Y \cup \{A\} \vdash}{Y \cup \{B\} \vdash}$ where $(A \leftrightarrow B) \in \mathsf{CPL}$.

Definition 4 (Proof of a sequent). A proof of a sequent $Y \vdash$ in $\mathsf{HI}^{\mathsf{CPL}}$ is a finite labelled tree regulated by the rules of $\mathsf{HI}^{\mathsf{CPL}}$, where the leaves are labelled with axioms and $Y \vdash$ labels the root.

A sequent $Y \vdash$ is provable in $\mathsf{HI}^{\mathsf{CPL}}$ iff the sequent $Y \vdash$ has at least one proof in $\mathsf{HI}^{\mathsf{CPL}}$.

Here are examples of proofs:

Example 1. A proof of the sequent $\neg(p \vee q), \neg(\neg p \wedge \neg q) \ \vdash$:

$$\neg p \wedge \neg q, p \ \vdash \qquad \neg p \wedge \neg q, q \ \vdash$$

$$\neg p \wedge \neg q, \neg\neg p \ \vdash \qquad \neg p \wedge \neg q, \neg\neg q \ \vdash$$

$$\neg p \wedge \neg q, \neg\neg p \vee \neg\neg q \ \vdash$$

$$\neg p \wedge \neg q . \neg(\neg p \wedge \neg q) \ \vdash$$

$$\neg(p \vee q), \neg(\neg p \wedge \neg q) \ \vdash$$

Example 2. A proof of the sequent $(p \vee q) \wedge \neg p, (p \vee q) \wedge \neg q, \neg p \wedge \neg q \ \vdash$:

$$q \wedge \neg p, p \wedge \neg q, \neg p \wedge \neg q \ \vdash$$

$$q \wedge \neg p, (q \wedge \neg q) \vee (p \wedge \neg q), \neg p \wedge \neg q \ \vdash$$

$$q \wedge \neg p, (p \vee q) \wedge \neg q, \neg p \wedge \neg q \ \vdash$$

$$(q \wedge \neg p) \vee (p \wedge \neg p), (p \vee q) \wedge \neg q, \neg p \wedge \neg q \ \vdash$$

$$(p \vee q) \wedge \neg p, (p \vee q) \wedge \neg q, \neg p \wedge \neg q \ \vdash$$

Provability of a sequent $Y \vdash$ yields that Y is HI-set. This is due to

Theorem 1 (Soundness of $\mathsf{HI}^{\mathsf{CPL}}$ w.r.t. HI-sets). *Let Y be an at least two element finite set of wffs. If the sequent $Y \vdash$ is provable in $\mathsf{HI}^{\mathsf{CPL}}$, then Y is a HI-set.*

Proof. Clearly, if $Y \vdash$ is an axiom, then Y is a HI-set.

Assume that $Y \cup \{A\}$ and $Y \cup \{B\}$ are HI-sets. Thus each wff in Y is consistent. Moreover, the set $Y \cup \{A \vee B\}$ is inconsistent – otherwise $Y \cup \{A\}$ would be consistent or $Y \cup \{B\}$ would be consistent. Suppose that the set $Y \cup \{A \vee B\}$ contains an inconsistent wff. Since each wff in Y is consistent, it follows that $A \vee B$ is inconsistent, and hence both A and B are inconsistent. But in this case neither $Y \cup \{A\}$ nor $Y \cup \{B\}$ is a HI-set. A contradiction.

It is obvious that if $X \cup \{A\}$ is a HI-set and $(A \leftrightarrow B) \in \mathsf{CPL}$, then $X \cup \{B\}$ is a HI-set. □

Theorem 1 together with corollaries 3, 2 and 1 yield:

Theorem 2.

1. *If the sequent $\neg C \vee p, \neg C \vee \neg p \vdash$ is provable in $\mathsf{HI}^{\mathsf{CPL}}$, then C is a valid wff.*
2. *If the sequent $C \vee p, C \vee \neg p \vdash$ is provable in $\mathsf{HI}^{\mathsf{CPL}}$, then C is an inconsistent wff.*
3. *If the sequent $C, \neg C \vdash$ is provable in $\mathsf{HI}^{\mathsf{CPL}}$, then C is a contingent wff.*

The next step is a non-standard one. We define provability of a wff in terms of provability of a sequent of a strictly defined form. But, contrary to what is usually done, we *do not* construe the provability of a wff C as the provability of the sequent based on C or the negation of C only. The definition runs as follows:

Definition 5 (Proof of a wff). A $\mathsf{HI}^{\mathsf{CPL}}$-proof of a wff C is a proof of the sequent $\neg C \vee p, \neg C \vee \neg p \vdash$ in $\mathsf{HI}^{\mathsf{CPL}}$.

Example 3. A proof of $p \rightarrow p$:

$$p, \neg p \vdash$$

$$(p \wedge \neg p) \vee p, \neg p \vdash$$

$$(p \wedge \neg p) \vee p, (p \wedge \neg p) \vee \neg p \vdash$$

$$\neg(p \rightarrow p) \vee p, (p \wedge \neg p) \vee \neg p \vdash$$

$$\neg(p \rightarrow p) \vee p, \neg(p \rightarrow p) \vee \neg p \vdash$$

Similarly, we define refutability in terms of provability of sequents of strictly defined form. This time, however, we introduce two concepts.

Definition 6 (Refutation[1] of a wff). A $\mathsf{HI}^{\mathsf{CPL}}$-refutation[1] of a wff C is a proof of the sequent $C \vee p, C \vee \neg p \vdash$ in $\mathsf{HI}^{\mathsf{CPL}}$.

Definition 7 (Refutation[2] of a wff). A $\mathsf{HI}^{\mathsf{CPL}}$-refutation[2] of a wff C is a proof of the sequent $C, \neg C \vdash$ in $\mathsf{HI}^{\mathsf{CPL}}$.

Example 4. A refutation[1] of $\neg(p \to p)$:

$$p, \neg p \vdash$$

$$\neg(\neg p \vee p) \vee p, \neg p \vdash$$

$$\neg(p \to p) \vee p, \neg p \vdash$$

$$\neg(p \to p) \vee p, \neg(\neg p \vee p) \vee \neg p \vdash$$

$$\neg(p \to p) \vee p, \neg(p \to p) \vee \neg p \vdash$$

Example 5. A refutation[2] of $p \to q$:

$$\neg p, p \wedge \neg q \vdash \qquad\qquad q, p \wedge \neg q \vdash$$

$$\neg p \vee q, p \wedge \neg q \vdash$$

$$p \to q, p \wedge \neg q \vdash$$

$$p \to q, \neg(p \to q) \vdash$$

The following holds:

Corollary 4.

1. *If C has a $\mathsf{HI}^{\mathsf{CPL}}$-proof, then C is valid.*
2. *If C has a $\mathsf{HI}^{\mathsf{CPL}}$-refutation[1], then C is inconsistent.*
3. *If C has a $\mathsf{HI}^{\mathsf{CPL}}$-refutation[2], then C is contingent.*

Proof. Immediately from Theorem 2 and definitions 5, 6, and 7. □

3.1 The completeness issue

The system $\mathsf{HI}^{\mathsf{CPL}}$ is complete with respect to finite HI-sets.

Theorem 3 (Completeness of $\mathsf{HI}^{\mathsf{CPL}}$ w.r.t. HI-sets). *Let Y be an at least two element finite set of wffs. If Y is a* HI*-set, then a sequent of the form $Y \vdash$ is provable in* $\mathsf{HI}^{\mathsf{CPL}}$.

Proof. Assume that $Y \vdash$ is in the normal form. Thus all the wffs in Y are in DNF.

By the *rank* of a sequent $Y \vdash$ (in symbols: $\mathbf{r}(Y \vdash)$) we mean the number of occurrences of the disjunction connective, $\vee$, in the wffs of Y.

Assume that Y is a finite HI-set. Note that Y has at least two elements.

Suppose that $\mathbf{r}(Y \vdash) = 0$. In this case, $Y \vdash$ is an axiom of the system.

Suppose that $\mathbf{r}(Y \vdash) > 0$. Let $\mathbf{r}(Y \vdash) = n$.

Inductive hypothesis. If $\mathbf{r}(X \vdash) < n$ and X is a HI-set of wffs in DNF, then the sequent $X \vdash$ is provable in $\mathsf{HI}^{\mathsf{CPL}}$.

If $\mathbf{r}(Y \vdash) = n$, where $n > 0$, the sequent $Y \vdash$ can be displayed as:

$$A_1, \ldots, A_{j-1}, B_1 \vee \ldots \vee B_k, A_{j+1}, \ldots, A_m \ \vdash$$

where $B_1, \ldots, B_k$ are elementary conjunctions and $k > 1$. As Y is a HI-set, at least one of $B_1, \ldots, B_k$ is consistent.

Let B_i be a consistent element of $\{B_1, \ldots, B_k\}$. Consider the sets Y' and Y'' defined by:

$$Y' = \{A_1, \ldots, A_{j-1}, B_i, A_{j+1}, \ldots, A_m\}$$

$$Y'' = \{A_1, \ldots, A_{j-1}, B_1 \vee \ldots \vee B_{i-1} \vee B_{i+1} \vee \ldots \vee B_k, A_{j+1}, \ldots, A_m\}$$

Clearly, $\mathbf{r}(Y' \vdash) < n$ and $\mathbf{r}(Y'' \vdash) < n$. Both $Y' \vdash$ and $Y'' \vdash$ are in the normal form.

If Y is a HI-set, so is Y'. Thus, by the inductive hypothesis, the sequent $Y' \vdash$ is provable.

As for the sequent $Y'' \vdash$, there are two cases to be considered.

Case 1. $B_1 \vee \ldots \vee B_{i-1} \vee B_{i+1} \vee \ldots \vee B_k$ is consistent. Thus Y'' is a HI-set. Hence, by the inductive hypothesis, the sequent $Y'' \vdash$ is provable. But one can get $Y \vdash$ from $Y' \vdash$ and $Y'' \vdash$ by an application of rule R_1 and then, if necessary, of rule R_2.

Case 2. $B_1 \vee \ldots \vee B_{i-1} \vee B_{i+1} \vee \ldots \vee B_k$ is inconsistent. Thus all the disjuncts (of the just considered disjunction) are inconsistent. If follows that B_i is CPL-equivalent to A_j. (Clearly we have $B_i \models A_j$. But, as all the disjuncts of $B_1 \vee \ldots \vee B_{i-1} \vee B_{i+1} \vee \ldots \vee B_k$ are inconsistent, their negations are valid and hence from $A_j \models B_1 \vee \ldots \vee B_k$ we get $A_j \models B_i$.) Thus one can get $Y \vdash$ from $Y' \vdash$ by R_2.

Now assume that $Y \vdash$ is not in the normal form. In order to complete the proof it suffices to observe that each CPL-wff is CPL-equivalent to a wff in DNF and thus one can always reach a wff from its DNF-counterpart by applying rule R_2. □

As a consequence of Theorem 3, Corollary 4, and definitions 5, 6, 7 one gets:

Theorem 4.

1. *A wff C is valid iff C has a* $\mathsf{HI}^{\mathsf{CPL}}$*-proof.*
2. *A wff C is inconsistent iff C has a* $\mathsf{HI}^{\mathsf{CPL}}$*-refutation*[1].
3. *A wff C is contingent iff C has a* $\mathsf{HI}^{\mathsf{CPL}}$*-refutation*[2].

3.2 Final remarks

The methodology used in the construction of the system $\mathsf{HI}^{\mathsf{CPL}}$, and the basic idea of the completeness proof, are very much alike to the methodology and idea applied, for different purposes, in Skura and Wiśniewski (2015).

As for this paper, the homogeneity effect has been achieved by using the notion of HI-set as a conceptual unifier. It is worth to note that the concept of minimally inconsistent set could have been used for this purpose as well. A set of wffs X is a minimally inconsistent set (MI-set for short) iff X is inconsistent, but each proper subset of X is consistent. When one deals with CPL, inconsistency, validity and contingency of wffs are expressible in terms of MI-sets as follows:

- A wff C is inconsistent iff $\{C\}$ is a MI-set.
- A wff C is valid iff $\{\neg C\}$ is a MI-set.
- A wff C is contingent iff $\{C, \neg C\}$ is a MI-set.

Thus once we have a system which "calculates" MI-sets, we get an alternative solution. A system of this kind exists (cf. Wiśniewski (2019)). The pros and cons issue remains to be studied.

The last remark is this. As for classical logic and some non-classical logics, one can define entailment by the clause:

(#) *X entails A iff the set* $X \cup \{\neg A\}$ *is inconsistent.*

However, a set of wffs can be inconsistent in different ways. One can differentiate between holistic inconsistency, minimal inconsistency, plain inconsistency, and so forth. Given this, one can then define different kinds of entailment, depending on the kind of inconsistency involved. In particular, if 'inconsistent' were replaced in (#) above with 'holistically inconsistent', we would get a non-Tarskian consequence relation with interesting properties. The system $\mathsf{HI}^{\mathsf{CPL}}$ offers a proof-theoretic account of entailment defined in this way (for the classical propositional case). However, this is another story.

References

Skura T (2009) A refutation theory. Logica Universalis 3(2):293–302

Skura T (2011) Refutation systems in propositional logic. In: Handbook of Philosophical Logic, Volume 16, Springer, pp 115–157

Skura T, Wiśniewski A (2015) A system for proper multiple-conclusion entailment. Logic and Logical Philosophy 24(2):241–253

Wiśniewski A (2019) Transmission of truth, entailment, and minimality. (submitted)