Erotetic calculi for Classical Propositional Logic and reduction to normal forms

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Abstract Some one-sided erotetic calculi for Classical Propositional Logic (hereafter: CPL) enable relatively easy conversions of a CPL-formula into equivalent CPL-formulas in Disjunctive Normal Form (DNF) and in Conjunctive Normal Form (CNF). A practical advantage of the proposed approach is the absence of steps which often lead to mistakes when "standard" conversion procedures are performed by humans, namely applying distributivity rules.

1 Preliminaries

By *wffs* we mean here CPL-formulas; the latter concept is defined in the usual manner. A *literal* is a propositional variable or the negation of a propositional variable. We use the letters A, B, possibly with subscripts, as metalanguage variables for wffs. The letters S, T, again possibly with subscripts, refer to sequences of wffs, the empty sequence included. We use the sign $'$ as the concatenation-sign for sequences of wffs. A metalanguage expression of the form $S \,'\, A$ denotes the concatenation of sequence S and the one-term sequence $\langle A \rangle$, while a metalanguage expression of the form $S \,'\, A \,'\, T$ refers to the concatenation of sequence $S \,'\, A$ and sequence T.

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We distinguish between α-wffs and β-wffs, according to the following table:

α	α_1	α_2	β	β_1	β_2
$A \wedge B$	A	B	$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	A	B
$\neg(A \rightarrow B)$	A	$\neg B$	$A \rightarrow B$	$\neg A$	B

We adopt the usual conventions concerning omitting brackets, in particular for DNFs and CNFs.

Let 1 stand for truth and 0 for falsity. A function v from the set of wffs to the set $\{1, 0\}$ is a CPL-*valuation* iff v satisfies the following conditions for any wffs A, B: (a) $v(\neg A) = 1$ iff $v(A) = 0$; (b) $v(A \vee B) = 1$ iff $v(A) = 1$ or $v(B) = 1$; (c) $v(A \wedge B) = 1$ iff $v(A) = 1$ and $v(B) = 1$; (d) $v(A \rightarrow B) = 1$ iff $v(A) = 0$ or $v(B) = 1$; (e) $v(A \leftrightarrow B) = 1$ iff $v(A) = v(B)$. Remark that the domain of a CPL-valuation is the whole set of wffs, and thus a CPL-valuation assigns either truth, 1, or falsity, 0, to each propositional variable.

In what follows we assume that the reader is familiar with the method of Socratic proofs.[1]

2 $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$ and DNF

In this section we consider left-sided erotetic calculus for CPL, resulting from the calculus $\mathbf{E}^{*}_{\mathsf{CPL}}$ (*cf.* [3]) by removing from it the rule R/L. We label the calculus considered by $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$. Rules of $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$ operate on questions which are based on sequences of left-sided sequents. A *left-sided sequent* is an expression of the form:

$$S \vdash \tag{1}$$

where S is a finite non-empty sequence of wffs, characterized by listing its consecutive terms. As for $\mathbf{E}^{*}_{\mathsf{CPL}}$ and, consequently, $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$, a *question* is an expression falling under the schema:

[1] A general introduction to the method can be found in Chapter 8 of [5]. Let me add that there exist many erotetic calculi for CPL and for First-Order Logic. Some non-classical logics (including normal modal propositional logics and intuitionistic propositional logic) have been formalized in the form of erotetic calculi as well. An interested reader is advised to consult, e.g., [1] or [2].

$$?(\Phi) \tag{2}$$

where Φ is a finite non-empty sequence of left-sided sequents.

In what follows, the letters Φ, Ψ will refer to sequences of sequents, and the semicolon, ';', performs the role of the concatenation-sign for sequences of sequents. We omit angle brackets when referring to a one-term sequence of sequents. Note that sequence $\langle S_1 \vdash, \ldots, S_m \vdash \rangle$ can be equivalently displayed as:[2]

$$S_1 \vdash; \ldots; S_m \vdash \tag{3}$$

We will be making use of this possibility below, in particular when displaying questions.

Here are the primary rules of the calculus $\mathbb{E}^{\circledast}_{\mathsf{CPL}}$:

$$\mathbb{E}^{\circledast}_{\alpha} : \quad \frac{?(\Phi;\ S\ '\ \alpha\ '\ T\ \vdash;\ \Psi)}{?(\Phi;\ S\ '\ \alpha_1\ '\ \alpha_2\ '\ T \vdash;\ \Psi)}$$

$$\mathbb{E}^{\circledast}_{\beta} : \quad \frac{?(\Phi;\ S\ '\ \beta\ '\ T \vdash;\ \Psi)}{?(\Phi;\ S\ '\ \beta_1\ '\ T \vdash;\ S\ '\ \beta_2\ '\ T \vdash;\ \Psi)}$$

$$\mathbb{E}^{\circledast}_{\neg\neg} : \quad \frac{?(\Phi;\ S\ '\ \neg\neg A\ '\ T \vdash;\ \Psi)}{?(\Phi;\ S\ '\ A\ '\ T \vdash;\ \Psi)}$$

Till the end of this section by a rule we will mean a primary rule of the calculus $\mathbb{E}^{\circledast}_{\mathsf{CPL}}$.

Let us recall:

Definition 1 (Socratic transformation)**.** *A sequence $\langle s_1, s_2, \ldots \rangle$ of questions is a Socratic transformation of a question $?(\Phi)$ via the rules of $\mathbb{E}^{\circledast}_{\mathsf{CPL}}$ iff the following conditions hold:*

1. $s_1 = ?(\Phi)$,
2. s_i, *where* $i > 1$, *results from* s_{i-1} *by an application of a rule of* $\mathbb{E}^{\circledast}_{\mathsf{CPL}}$.

We introduce the following auxiliary concept:

[2] Since $\langle S_1 \vdash, \ldots, S_m \vdash \rangle = \langle S_1 \vdash \rangle; \ldots; \langle S_m \vdash \rangle$, and angle brackets can be omitted in the case of one-term sequences of sequents.

Definition 2 (Dissatisfaction of a left-sided sequent)**.** *A* CPL*-valuation* v *dissatisfies a sequent* $S \vdash$ *iff* $v(A) = 0$ *for some term* A *of* S*; otherwise we say that* v *satisfies the sequent* $S \vdash$.

Rules of $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$ preserve dissatisfaction of left-sided sequents in both directions. One can prove:

Lemma 1. *Assume that question* $?(\Psi)$ *results from question* $?(\Phi)$ *by a rule of* $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$*. Let* v *be a* CPL*-valuation. Then* v *dissatisfies each sequent of* Ψ *iff* v *dissatisfies each sequent of* Φ.

Proof. By cases. □

Lemma 2. *Let* s *be a finite Socratic transformation of a question of the form* $?(A \vdash)$ *via the rules of* $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$*, and let* v *be a* CPL*-valuation. Then* $v(A) = 0$ *iff* v *dissatisfies each sequent that occurs in the last term/question of* s.

Proof. By Lemma 1. □

Let us introduce:

Definition 3 (Completed Socratic transformation)**.** *A Socratic transformation of a question via the rules of an erotetic calculus is completed iff the transformation is finite and no rule of the erotetic calculus is applicable to the last term/question of it.*

If A is a literal, then the one-term sequence, $\langle ?(A \vdash) \rangle$, is the only completed Socratic transformation of the question $?(A \vdash)$ via the rules of $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$. If A is not a literal, a completed Socratic transformation of the corresponding question of the form $?(A \vdash)$ via the rules of $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$ has at least two terms. The rules of $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$ are eliminative w.r.t. occurrences of binary propositional connectives and double negations. A Socratic transformation via the rules of $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$ develops by acting upon the question arrived at in the previous step (or upon the initial question in the first step). Thus any Socratic transformation via the rules of $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$ is finite. Moreover, the following holds:

Corollary 1. *For each question of the form* $?(A \vdash)$ *there exists a completed Socratic transformation of the question via the rules of* $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$.

Corollary 2. *Each completed Socratic transformation, via the rules of* $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$*, of a question of the form* $?(A \vdash)$ *ends with a question having the following property:*

($\heartsuit$) *every sequent that occurs in the last question of the transformation involves only literals.*[3]

We need:

Definition 4. *Let*

$$\langle B_1, \ldots, B_n \rangle$$

be a finite non-empty sequence of literals.

$$\mathsf{f}(\langle B_1, \ldots, B_n \rangle) = \begin{cases} B_1 \text{ if } n = 1 \\ (B_1 \wedge \ldots \wedge B_n) \text{ if } n > 1 \end{cases}$$

Definition 5. *Let*

$$\langle S_1 \vdash, \ldots, S_m \vdash \rangle$$

be a finite non-empty sequence of left-sided sequents such that $S_1, \ldots, S_m$ *are non-empty finite sequences of literals.*

$$\mathsf{g}(\langle S_1 \vdash, \ldots, S_m \vdash \rangle) = \begin{cases} \mathsf{f}(S_1) \text{ if } m = 1 \\ \mathsf{f}(S_1) \vee \ldots \vee \mathsf{f}(S_m) \text{ if } m > 1 \end{cases}$$

The following holds:

Corollary 3. *If*

$$\langle S_1 \vdash, \ldots, S_m \vdash \rangle$$

is a finite non-empty sequence of left-sided sequents such that each sequent of the sequence involves only literals, then

$$\mathsf{g}(\langle S_1 \vdash, \ldots, S_m \vdash \rangle)$$

is a wff in DNF.

Let us now prove:

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[3] More precisely: each wff which is a term of the sequence of wffs that occurs in a sequent is a literal.

Theorem 1. *Let* s *be a completed Socratic transformation of a question of the form* $?(A \vdash)$ *via the rules of* $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$*, and let:*

$$?(S_1 \vdash; \ldots; S_m \vdash) \tag{4}$$

be the last term of s*. The following condition holds:*

– *for each* CPL*-valuation* v*:*

$$v(A) = 1 \textit{ iff } v(\mathsf{g}(\langle S_1 \vdash, \ldots, S_m \vdash\rangle)) = 1.$$

Proof. Recall that

$$S_1 \vdash; \ldots; S_m \vdash$$

equals $\langle S_1 \vdash, \ldots, S_m \vdash\rangle$.

($\Rightarrow$). Assume that $v(A) = 1$. Thus, by Lemma 2, v satisfies some sequent of the sequence:

$$\langle S_1 \vdash, \ldots, S_m \vdash\rangle \tag{5}$$

It follows that for some sequent, $S_k \vdash$, of the sequence (5) we have $\mathsf{f}(S_k \vdash) = 1$ and hence $v(\mathsf{g}(\langle S_1 \vdash, \ldots, S_m \vdash\rangle)) = 1$.

($\Leftarrow$) Assume that $v(\mathsf{g}(\langle S_1 \vdash, \ldots, S_m \vdash\rangle)) = 1$. Hence it is not the case that v dissatisfies all the sequents of the sequence (5). Thus, by Lemma 2, $v(A) \neq 0$. Hence $v(A) = 1$. □

Obviously, the following holds:

Lemma 3. $(A \leftrightarrow B) \in \mathsf{CPL}$ *just in case for each* CPL*-valuation* v*:* $v(A) = 1$ *iff* $v(B) = 1$.

Finally, we get:

Theorem 2. *Let* s *be a completed Socratic transformation of a question of the form* $?(A \vdash)$ *via the rules of* $\mathbf{E}^{\circledast}_{\mathsf{CPL}}$*, and let:*

$$?(S_1 \vdash; \ldots; S_m \vdash)$$

be the last term of s*. Then:*

– $\mathsf{g}(\langle S_1 \vdash, \ldots, S_m \vdash\rangle)$ *is a wff in DNF,*
– $(A \leftrightarrow \mathsf{g}(\langle S_1 \vdash, \ldots, S_m \vdash\rangle)) \in \mathsf{CPL}$.

Proof. By Theorem 1 and Corollary 3. □

Thus in order to convert a CPL-formula, A, into an equivalent formula in DNF it suffices to perform a completed Socratic transformation of the question $?(A \vdash)$ in the erotetic calculus $\mathbb{E}^{\circledast}_{\mathsf{CPL}}$.

Example 1.

$$?((p \to q) \land p \to q \vdash)$$

$$?(\neg((p \to q) \land p) \vdash; q \vdash)$$

$$?(\neg(p \to q) \vdash; \neg p \vdash; q \vdash)$$

$$?(p, \neg q \vdash; \neg p \vdash; q \vdash)$$

a DNF of $(p \to q) \land p \to q$ is:

$$(p \land \neg q) \lor \neg p \lor q \tag{6}$$

as

$$p, \neg q \vdash; \neg p \vdash; q \vdash$$

equals

$$\langle p, \neg q \vdash, \neg p \vdash, q \vdash \rangle$$

and $\mathsf{g}(\langle p, \neg q \vdash, \neg p \vdash, q \vdash \rangle)$ is $(p \land \neg q) \lor \neg p \lor q$.

3 $\mathbb{E}^{**}_{\mathsf{CPL}}$ and CNF

Let us now turn to the right-sided erotetic calculus for CPL (*cf*. [4]), which we will label here $\mathbb{E}^{**}_{\mathsf{CPL}}$. Rules of the system operate on questions based on sequences of right-sided sequents. A *right-sided sequent* has the form:

$$\vdash S \tag{7}$$

where S is a finite non-empty sequence of wffs, again characterized by listing its consecutive terms. As for questions, the presence of right-sided sequents instead of left-sided sequents makes the only difference; all the conventions introduced in the previous sections apply.

Here are the primary rules of the calculus $\mathbb{E}^{**}_{\mathsf{CPL}}$:

$$\mathbf{E}^{**}_{\alpha} : \frac{?(\Phi;\ \vdash S\ '\ \alpha\ '\ T;\ \Psi)}{?(\Phi;\ \vdash S\ '\ \alpha_1\ '\ T;\ \vdash S\ '\ \alpha_2\ '\ T;\ \Psi)}$$

$$\mathbf{E}^{**}_{\beta} : \frac{?(\Phi;\ \vdash S\ '\ \beta\ '\ T;\ \Psi)}{?(\Phi;\ \vdash S\ '\ \beta_1\ '\ \beta_2\ '\ T;\ \Psi)}$$

$$\mathbf{E}^{**}_{\neg\neg} : \frac{?(\Phi;\ \vdash S\ '\ \neg\neg A\ '\ T;\ \Psi)}{?(\Phi;\ \vdash S\ '\ A\ '\ T;\ \Psi)}$$

In what follows by a rule of $\mathbf{E}^{**}_{\mathsf{CPL}}$ we mean a primary rule of the system. We introduce the following auxiliary concept:

Definition 6 (Satisfaction of a right-sided sequent)**.** *A* CPL*-valuation* v *satisfies a sequent* $\vdash S$ *iff* $v(A) = 1$ *for some term* A *of* S*.*

One can prove:

Lemma 4. *Assume that question* $?(\Psi)$ *results from question* $?(\Phi)$ *by a rule of* $\mathbf{E}^{**}_{\mathsf{CPL}}$*. Let* v *be a* CPL*-valuation. Then* v *satisfies each sequent of* Ψ *iff* v *satisfies each sequent of* Φ*.*

Proof. By cases. □

Thus rules of $\mathbf{E}^{**}_{\mathsf{CPL}}$ preserve satisfaction of right-sided sequents (understood in the sense of Definition 6) in both directions. As a consequence we get:

Lemma 5. *Let* s *be a Socratic transformation of a question of the form* $?(\vdash A)$ *via the rules of* $\mathbf{E}^{**}_{\mathsf{CPL}}$*, and let* v *be a* CPL*-valuation. Then* $v(A) = 1$ *iff* v *satisfies each sequent that occurs in the last question of* s*.*

Proof. By Lemma 4. □

The concept of a completed Socratic transformation of a question of the form $?(\vdash A)$ via the rules of $\mathbf{E}^{**}_{\mathsf{CPL}}$ is defined accordingly. Analogously as before, one can easily prove:

Corollary 4. *For each question of the form* $?(\vdash A)$ *there exists a completed Socratic transformation of the question via the rules of* $\mathbf{E}^{**}_{\mathsf{CPL}}$*.*

Corollary 5. *Each sequent which occurs in the last question of a completed Socratic transformation, via the rules of* $\mathbf{E}^{**}_{\mathsf{CPL}}$*, of a question of the form* $?(\vdash A)$ *involves only literals.*

Let us introduce:

Definition 7. *Let*

$$\langle B_1, \ldots, B_n \rangle$$

be a finite non-empty sequence of literals.

$$\mathsf{h}(\langle B_1, \ldots, B_n \rangle) = \begin{cases} B_1 \textit{ if } n = 1 \\ (B_1 \vee \ldots \vee B_n) \textit{ if } n > 1 \end{cases}$$

Definition 8. *Let*

$$\langle \vdash S_1, \ldots, \vdash S_m \rangle$$

be a finite non-empty sequence of right-sided sequents such that $S_1, \ldots, S_m$ *are sequences of literals.*

$$\mathsf{k}(\langle \vdash S_1, \ldots, \vdash S_m \rangle) = \begin{cases} \mathsf{h}(S_1) \textit{ if } m = 1 \\ \mathsf{h}(S_1) \wedge \ldots \wedge \mathsf{h}(S_m) \textit{ if } m > 1 \end{cases}$$

We get:

Corollary 6. *If*

$$\langle \vdash S_1, \ldots, \vdash S_m \rangle$$

is a finite non-empty sequence of right-sided sequents such that each sequent of the sequence involves only literals, then

$$\mathsf{k}(\langle \vdash S_1, \ldots, \vdash S_m \rangle)$$

is a wff in CNF.

Our next theorem presents a result, in a sense, parallel to that expressed by Theorem 1:

Theorem 3. *Let* s *be a completed Socratic transformation of a question of the form* $?(\vdash A)$ *via the rules of* $\mathbb{E}^{**}_{\mathsf{CPL}}$*, and let:*

$$?(\vdash S_1; \ldots; \vdash S_m) \tag{8}$$

be the last term of s*. The following condition holds:*

– *for each* CPL*-valuation* v*:*

$$v(A) = 1 \textit{ iff } v(\mathsf{k}(\langle \vdash S_1, \ldots, \vdash S_m \rangle)) = 1.$$

Proof. Similar to that of Theorem 1 (we apply Lemma 5 instead of Lemma 2). □

As a consequence we get:

Theorem 4. *Let* s *be a completed Socratic transformation of a question of the form* $?(\vdash A)$ *via the rules of* $\mathbf{E}^{**}_{\mathsf{CPL}}$*, and let:*

$$?(\vdash S_1; \ldots; \vdash S_m)$$

be the last term of s*. Then:*

- $\mathsf{k}(\langle \vdash S_1; \ldots; \vdash S_m \rangle)$ *is a wff in CNF,*
- $(A \leftrightarrow \mathsf{k}(\langle \vdash S_1; \ldots; \vdash S_m \rangle)) \in \mathsf{CPL}$.

Hence in order to find a CNF of a CPL-wff, A, which is not already in CNF it suffices to perform a completed Socratic transformation of the question $?(\vdash A)$ in the erotetic calculus $\mathbf{E}^{**}_{\mathsf{CPL}}$.

Example 2.

$$?(\vdash (p \to q) \land p \to q)$$

$$?(\vdash \neg((p \to q) \land p), q)$$

$$?(\vdash \neg(p \to q), \neg p, q)$$

$$?(\vdash p, \neg p, q; \vdash \neg q, \neg p, q)$$

Since

$$\vdash p, \neg p, q; \vdash \neg q, \neg p, q$$

is

$$\langle \vdash p, \neg p, q, \vdash \neg q, \neg p, q \rangle$$

and

$$\mathsf{k}(\langle \vdash p, \neg p, q, \vdash \neg q, \neg p, q \rangle)$$

equals

$$(p \lor \neg p \lor q) \land (\neg q \lor \neg p \lor q) \tag{9}$$

formula (9) is a CNF of the wff:

$$(p \to q) \land p \to q \tag{10}$$

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