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Two Logics of Occurrent Belief*

1. This paper presents two systems, $\Sigma.0$ and $\Sigma.1$, of modal logic which, from the formal point of view, are contained in the Łukasiewicz modal system **Ł**. These systems may be interpreted as epistemic logics. We argue that each of the systems $\Sigma.0$ and $\Sigma.1$ may be regarded as a logic of *occurrent belief and admitting*. If $\Sigma.0$ and $\Sigma.1$ are viewed from this perspective, the difference between them lies in the fact that $\Sigma.1$ imposes stronger conditions of rationality on the (epistemic) agent in question.

We shall start from the presentation of the system $\Sigma.0$ and then we shall discuss its proposed epistemic interpretation. This discussion will lead us to the system $\Sigma.1$.

Let us add that the system $\Sigma.0$ was constructed for different reasons (cf. Wiśniewski 1995) and thus also has other epistemic interpretations.

2. Both $\Sigma.0$ and $\Sigma.1$ are based on Classical Propositional Calculus (CPC) and are worded in the same formal language $\mathcal{L}_{\Sigma.0}$. The language $\mathcal{L}_{\Sigma.0}$ contains the propositional variables $p, q, r, \ldots$, the classical connectives $\neg$ (negation), $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), $\leftrightarrow$ (equivalence), brackets (,), and two modal operators: $\Box$ and $\Diamond$.[1] The set of well-formed formulas of $\mathcal{L}_{\Sigma.0}$ (wffs for short) is the smallest set which contains all propositional variables and fulfils the following conditions: (i) if A is a wff of $\mathcal{L}_{\Sigma.0}$, then $\neg A$, $\Box A$ and $\Diamond A$ are wffs of $\mathcal{L}_{\Sigma.0}$; (ii) if A, B are wffs of $\mathcal{L}_{\Sigma.0}$, then $(A \rightarrow B), (A \wedge B), (A \vee B), (A \leftrightarrow B)$ are wffs of $\mathcal{L}_{\Sigma.0}$. We will use the letters A, B, C, E as metalinguistic variables for wffs of $\mathcal{L}_{\Sigma.0}$. We adopt here the usual conventions concerning omitting brackets.

[1] In the 1998 version of this paper the signs S and D are used instead. The symbol S alludes to "stwierdzanie" and D to "dopuszczanie", the propositional attitudes analysed in Wiśniewski (1995).

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Let *Taut* be the set of all wffs of $\mathcal{L}_{\Sigma.0}$ which result from the theses of CPC by uniformly replacing the propositional variables by wffs of $\mathcal{L}_{\Sigma.0}$. Axioms of the system $\Sigma.0$ are wffs of $\mathcal{L}_{\Sigma.0}$ of the following forms:

Ax.0. A, where $A \in$ *Taut*,

Ax.1. $\Box(A \to B) \to (\Box A \to \Box B)$,

Ax.2. $\Box(A \wedge B) \to \Box A \wedge \Box B$,

Ax.3. $\Box A \wedge \Box B \to \Box(A \wedge B)$,

Ax.4. $\Box A \to \Diamond A$,

Ax.5. $\Box A \to \neg\Diamond\neg A$,

Ax.6. $\Diamond A \to \neg\Box\neg A$.

The only rule of inference of $\Sigma.0$ is *Modus Ponens* (MP):

$$\frac{\begin{array}{c} A \to B, \\ A \end{array}}{B}$$

One remark is in order here: contrary to appearance, the axioms Ax.2 and Ax.3 are not superfluous. The system $\Sigma.0$ does not have theorems of the form $\Box A$ (see below); thus $\Box A$ is not a thesis of $\Sigma.0$ even if $A \in$ *Taut* and hence the axioms Ax.2 and Ax.3 cannot be derived from the axiom Ax.1 in the usual way.

One can prove that the following are (schemes of) theorems of the system $\Sigma.0$:

(T1) $\Box(A \wedge B) \to \Box A$,

(T2) $\Box(A \wedge B) \to \Box B$,

(T3) $\Box A \to \neg\Box\neg A$,

(T4) $\Box\neg A \to \neg\Box A$,

(T5) $\neg\Box(A \wedge \neg A)$,

(T6) $\Box A \wedge \Box(A \to B) \to \Box B$,

(T7) $\Box(A \to B) \wedge \Box(B \to C) \to (\Box A \to \Box C)$,

(T8) $\Box(A \vee B) \wedge \neg\Box(A \wedge B) \to \neg\Box A \vee \neg\Box B$,

(T9) $\Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$, (T10) $\Box(A \to B) \wedge \neg\Box B \to \neg\Box A$,

(T11) $\Box(A \to B) \wedge \Box\neg B \to \neg\Box A$,

(T12) $\Box(A \to B) \wedge \Box A \to \neg\Diamond\neg B$,

(T13) $\Box(A \to (B \to C)) \wedge \Box A \wedge \Box B \to \Box C$,

(T14) $\neg\Diamond A \to \neg\Box A$,

(T15) $\Diamond\neg A \to \neg\Box A$,

(T16) $\Box\neg A \to \neg\Diamond A$.

One can easily find further examples of this kind.

3. Let us now examine some properties of the system $\Sigma.0$.

Let:

$$\mathfrak{M}_{\text{Ł}} = \langle\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{0}\}, f_{\neg}, f_{\to}, f_{\wedge}, f_{\vee}, f_{\leftrightarrow}, f_{\Box}, f_{\Diamond}, \{\mathbf{1}\}\rangle$$

be a four-valued matrix (with **1** as the designated value) such that $f_{\neg}$, $f_{\to}$, $f_{\wedge}$, $f_{\vee}$, $f_{\leftrightarrow}$, $f_{\Box}$, $f_{\Diamond}$ are defined by the tables:

	$f_{\neg}$
1	**0**
2	**3**
3	**2**
0	**1**

$f_{\to}$	**1**	**2**	**3**	**0**
1	**1**	**2**	**3**	**0**
2	**1**	**1**	**3**	**3**
3	**1**	**2**	**1**	**2**
0	**1**	**1**	**1**	**1**

$f_{\wedge}$	**1**	**2**	**3**	**0**
1	**1**	**2**	**3**	**0**
2	**2**	**2**	**0**	**0**
3	**3**	**0**	**3**	**0**
0	**0**	**0**	**0**	**0**

$f_{\vee}$	**1**	**2**	**3**	**0**
1	**1**	**1**	**1**	**1**
2	**1**	**2**	**1**	**2**
3	**1**	**1**	**3**	**3**
0	**1**	**2**	**3**	**0**

$f_{\leftrightarrow}$	**1**	**2**	**3**	**0**
1	**1**	**2**	**3**	**0**
2	**2**	**1**	**0**	**3**
3	**3**	**0**	**1**	**2**
0	**0**	**3**	**2**	**1**

	$f_{\Box}$
1	**2**
2	**2**
3	**0**
0	**0**

	$f_{\Diamond}$
1	**1**
2	**1**
3	**3**
0	**3**

One can verify the following:

COROLLARY 1. *Each theorem of the system* $\Sigma.0$ *is satisfied by the matrix* $\mathfrak{M}_{\text{Ł}}$.

The matrix $\mathfrak{M}_{Ł}$ is an adequate matrix for the Łukasiewicz modal system **Ł**.[2] Thus, from the purely formal point of view, the system $\Sigma.0$ is included in Łukasiewicz's system **Ł**.

As an immediate consequence of Corollary 1 we get:

FACT 1. *The system* $\Sigma.0$ *has no theorems of the form* $\Box A$.

It follows that the analogue of rule of necessitation is not derivable in $\Sigma.0$ and that $\Sigma.0$ is not closed under this rule.

Let:

$$\mathfrak{M}^* = \langle \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{0}\}, f_\neg, f_\rightarrow, f_\wedge, f_\vee, f_\leftrightarrow, f^*_\Box, f^*_\Diamond, \{\mathbf{1}\}\rangle$$

be a four-valued matrix (with **1** as the designated value) such that $f_\neg$, $f_\rightarrow$, $f_\wedge$, $f_\vee$, $f_\leftrightarrow$ are defined as above, but $f^*_\Box$, $f^*_\Diamond$ are defined by the tables:

	$f^*_\Box$
1	**3**
2	**3**
3	**0**
0	**0**

	$f^*_\Diamond$
1	**3**
2	**3**
3	**2**
0	**0**

We can verify the following

COROLLARY 2. *Each theorem of the system* $\Sigma.0$ *is satisfied by the matrix* $\mathfrak{M}^*$.

This result has some interesting consequences. First, we have:

FACT 2. *The system* $\Sigma.0$ *has no theorems of the form* $\Diamond A$.

Thus, looking from the formal point of view, the system $\Sigma.0$ is a subsystem of the system **Ł**; the latter contains, inter alia, theorems of the form $\mathsf{M}A$, where M stands for "it is possible that".

One can show that the following wffs of $\mathcal{L}_{\Sigma.0}$ are not satisfied by the matrix $\mathfrak{M}^*$.

(1) $\Box p \vee \Box\neg p$,

(2) $\Box p \rightarrow p$,

(3) $p \rightarrow \Box p$,

(4) $p \rightarrow \Diamond p$,

[2] Cf. Łukasiewicz (1957), Smiley (1961), (1963).

(5) $\Diamond p \rightarrow p$,

(6) $\Diamond p \vee \Diamond\neg p$,

(7) $\neg\Diamond\neg p \rightarrow \Box p$,

(8) $\neg\Box\neg p \rightarrow \Diamond p$,

Thus the wffs (1) – (8) are not theorems of the system $\Sigma.0$. Hence the following:

(9) $\Box A \leftrightarrow \neg\Diamond\neg A$,

(10) $\Diamond A \leftrightarrow \neg\Box\neg A$

are not (schemes of) theorems of $\Sigma.0$. As a result we get

FACT 3. *The modal operators $\Box$ and $\Diamond$ are not interdefinable in the system* $\Sigma.0$.

By means of the matrix $\mathfrak{M}^*$, one can also show that the following are not theorems of the system $\Sigma.0$:

(11) $\Box\Box p \leftrightarrow \Box p$,

(12) $\Diamond\Diamond p \leftrightarrow \Diamond p$,

(13) $\Box\Diamond p \leftrightarrow \Diamond\Box p$,

(14) $\Box\Diamond p \leftrightarrow \Diamond p$,

(15) $\Diamond\Box p \leftrightarrow \Box p$.

Thus the corresponding modalities are distinct in $\Sigma.0$.

We do not claim here, however, that the matrix $\mathfrak{M}^*$ is an adequate matrix for the system $\Sigma.0$.

4. The modal operators $\Box$ and $\Diamond$ may be interpreted in many ways. One possible interpretation – and a rather intuitive one – is to understand the operator $\Box$ as referring to *occurrent belief* and the operator $\Diamond$ as referring to *occurrent admitting*. To be more precise, we can read the operator $\Box$ as "person *x actively believes* that", and the operator $\Diamond$ as "person *x actively admits* that", where the person *x* is assumed to meet some conditions of rationality (see below).

Let us observe that if the above reading of $\Box$ and $\Diamond$ is taken for granted, then the properties of $\Sigma.0$ can be explained in a natural way. In particular, this reading explains why $\Box$ and $\Diamond$ should not be interdefinable and why the formulas (1) – (8) are not theorems of $\Sigma.0$. Let us start with the interdefinability problem. Looking from the intuitive point of view, it may happen that a sentence of the form "it is not the case that person x actively admits that $\neg p$" is true for the reason that x is unaware of both p and $\neg p$; in this situation the sentence "person x actively believes that p" is false. Hence the formula (7) should not be a theorem of $\Sigma.0$ – and in fact it isn't. Similarly, if it is not the case that person x actively believes that $\neg p$, then it need not be the case that person x actively admits that p – it may happen that person x is unaware of both p and $\neg p$. Thus (8) need not be a theorem of $\Sigma.0$ – and in fact it isn't. For the same reason (1) and (6) should not be theorems of $\Sigma.0$. Both belief and admitting are not factive – thus (2) and (5) should not be theorems of $\Sigma.0$. And finally, person x may neither actively believe nor actively admit that p even if p is true, so (3) and (4) cannot be theorems of $\Sigma.0$.

The system $\Sigma.0$ does not contain theorems of the form $\Box A$ and of the form $\Diamond A$. But if $\Box$ is understood as referring to occurrent belief, there are no reasons for which *logic* should force an occurrent belief concerning any proposition – even if this proposition belongs to logic itself! It may be true or false that some person actively believes that something holds, but this is a matter of facts, not of logic. The situation is similar in the case of $\Diamond$. Under the analysed interpretation, $\Diamond$ refers to occurrent admitting. But there are no reasons for which logic should prejudge an occurrent admitting concerning any proposition.

So far we have discussed the "negative" properties of $\Sigma.0$. But its "positive" properties also become natural when the above interpretation of $\Box$ and $\Diamond$ is taken into account. Let us start with the discussion of the axioms.

From the intuitive point of view, the axioms Ax.4, Ax.5 and Ax.6 are now acceptable without any qualifications.

The axiom Ax.2 says that occurrent belief distributes over conjunction. This is not always the case, but if the person in question is rational, it must be the case. So the axiom Ax.2 is acceptable if the epistemic agent is rational. The situation is similar in the case of the axiom Ax.3. We may also say that Ax.3 assigns to an agent of occurrent belief a kind of "automatic deductive ability": occurrent belief that conjuncts hold immediately yields occurrent belief that the whole conjunction holds. The axiom Ax.2, in turn, may be regarded as assigning to an epistemic agent some "analytical abilities."

Let us now consider the axiom Ax.1. Again, it may happen that some person actively believes that an implication holds and its antecedent holds, but does not actively believe that the consequent holds. Yet, such a situation is impossible if the person in question is to be called rational. So the axiom

Ax.1 is acceptable with the same qualifications as axioms Ax.2 and Ax.3. Let us stress that the axiom Ax.1 can also be interpreted as assigning to an (epistemic) agent a kind of "deductive ability": it says that occurrent belief that an implication holds transforms into occurrent belief that its consequent is the case when the antecedent of this implication happens to be actively believed.

The axiom Ax.1 guarantees, inter alia, that when a valid implication (e.g. a classical tautology) is actively believed and its antecedent is actively believed, then the consequent of this implication is actively believed as well. This does not mean, however, that $\Sigma.0$ yields that occurrent beliefs are closed under all valid implications (i.e. that an agent actively believes all the logical consequences of his/her occurrent beliefs). Occurrent beliefs concerning valid implications are not warranted by $\Sigma.0$ – there are no theorems of the form $\Box(A \to B)$, where $A \to B$ is a valid implication (cf. above, Fact 1). It may also be proved that $\Sigma.0$ has the following property:

FACT 4. *There are wffs* A, B *of* $\mathcal{L}_{\Sigma.0}$ *such that* $\ulcorner A \to B \urcorner \in$ *Taut and the wff* $\Box A \to \Box B$ *is not a theorem of* $\Sigma.0$.

For the proof see Świrydowicz (1995); this paper also includes a different interpretation of the theorems of $\Sigma.0$, as well as the construction of a Rantala-style semantics for $\Sigma.0$. Yet, looking from our point of view, the result expressed above means that occurrent belief as described by $\Sigma.0$ is not closed under valid implications – a result which gives us an argument for the plausibility of our epistemic interpretation of $\Sigma.0$.

5. The axioms and theses of $\Sigma.0$ may be regarded as describing the properties of occurrent belief of a rational agent. Let us observe, however, that our rational agent can also be quite "irrational" in some respects. It may happen that our epistemic agent actively believes that a certain proposition holds, but does not actively believe that some immediate consequence(s) of this proposition hold(s) as well. In particular, it is not excluded that an epistemic agent actively believes that A, but at the same time does not (actively) believe that A^*, where A^* is equivalent with A, but contains different propositional connectives. For example, $\Sigma.0$ allows that a person x actively believes that a proposition of the form $A \lor B$ holds without believing that $\neg A \to B$ holds, or that x actively believes that $A \to ((B \to C) \land (C \to B))$ and does not actively believe that $A \to (B \leftrightarrow C)$. Although such consequences are realistic (unfortunately, as a teacher of logic might say), we may want to avoid them. For example, we may assume that the rational occurrent beliefs should be closed under the rule of definitional replacement with respect to the definitions of some propositional connectives in terms of others (e.g. definitions of conjunction, disjunction and equivalence by means of implication and negation). Our second system, that is, the system $\Sigma.1$, complies with this intuition.

6. The system $\Sigma.1$ has the same axioms as $\Sigma.0$. The difference lies in the fact that the system $\Sigma.1$ has one additional primary rule of inference. To be more precise, the rules of inference of $\Sigma.1$ are *Modus Ponens* (MP) and the following rule of definitional replacement (DR):

$$\frac{\begin{array}{c} A =_{df} B, \\ C \end{array}}{C[A/B]}$$

where the relevant definitions are

Df.1 $A \wedge B =_{df} \neg(A \to \neg B)$,

Df.2 $A \vee B =_{df} \neg A \to B$,

Df.3 $A \leftrightarrow B =_{df} \neg((A \to B) \to \neg(B \to A))$.

Note that the rule (DR) is not redundant here: although the elements of *Taut* (i.e. the propositional tautologies) are axioms of $\Sigma.1$, the rule (DR) allows us to replace wffs which occur in the scope of a modal operator. Let us recall here that the basic system $\Sigma.0$ is not closed under the rule of regularity (cf. above, Fact 4) and also under the rule of extensionality (cf. Świrydowicz 1995).

It is clear that each theorem of $\Sigma.0$ is also a theorem of $\Sigma.1$. We may prove that the following are (schemes of) theorems of $\Sigma.1$:

(T16) $\Box(A \leftrightarrow B) \to \Box(A \to B)$,

(T17) $\Box(A \leftrightarrow B) \to \Box(B \to A)$,

(T18) $\Box(A \to B) \wedge \Box(B \to A) \to \Box(A \leftrightarrow B)$,

(T19) $\Box(A \leftrightarrow B) \leftrightarrow \Box((A \to B) \wedge (B \to A))$,

(T20) $\Box(A \leftrightarrow B) \to (\Box A \leftrightarrow \Box B)$,

(T21) $\Box\neg(A \to \neg B) \to \neg(\Box A \to \neg\Box B)$,

(T22) $\neg(\Box A \to \neg\Box B) \to \Box\neg(A \to \neg B)$,

(T23) $\Box(A \vee B) \wedge \Box\neg A \to \Box B$,

(T24) $\Box(A \vee B) \to \neg\Box(\neg A \wedge \neg B)$.

It is easy to find further examples of this kind.

7. From the formal point of view, the system $\Sigma.1$ is also contained in the Łukasiewicz modal system **Ł**. One can easily prove

COROLLARY 3. *Each theorem of the system* $\Sigma.1$ *is satisfied by the matrix* $\mathfrak{M}_{Ł}$.

As a result we get

FACT 5. *The system* $\Sigma.1$ *has no theorems of the form* $\Box A$.

We can also prove

COROLLARY 4. *Each theorem of the system* $\Sigma.1$ *is satisfied by the matrix* $\mathfrak{M}^*$.

FACT 6. *The system* $\Sigma.1$ *has no theorems of the form* $\Diamond A$.

Of course, the formulas (1) – (8) are also not theorems of $\Sigma.1$; so $\Box$ and $\Diamond$ are not interdefinable in $\Sigma.1$.

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Weak Epistemic Logic, Immediate Consequence, and Paraconsistency*

1. Introduction. In Wiśniewski (1995) a system of modal logic, called $\Sigma.0$, was introduced. A semantical analysis of $\Sigma.0$ and some of its subsystems was then given by Świrydowicz (1995); the analysis is carried out in terms of possible worlds semantics and by means of, inter alia, non-normal possible worlds. The system $\Sigma.0$ was interpreted in Wiśniewski (1998) as an epistemic logic, namely, as a logic of occurrent belief.

$\Sigma.0$ is built in a language which results from the language of Classical Propositional Calculus (CPC) by adding modal operators. To be more precise, the vocabulary of the language contains propositional variables p, q, r, ..., the connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (material equivalence), two modal operators[1]: $\Box$, $\Diamond$, and brackets. Well-formed formulas (wffs for short) are defined as usual; iterations of modal operators are permitted. We shall designate the language described above by $\mathcal{L}_{\Sigma.0}$ and we use the letters $A, B, C, \ldots$ (with subscripts if needed) as metalinguistic variables for wffs of $\mathcal{L}_{\Sigma.0}$. The Greek lower case letters $\alpha, \beta, \gamma, \ldots$ will be used as metalinguistic variables for CPC-formulas. By *Taut* we shall designate the set of wffs of $\mathcal{L}_{\Sigma.0}$ whose elements result from CPC-valid formulas by uniformly replacing the propositional variables by wffs of $\mathcal{L}_{\Sigma.0}$.

Axioms of $\Sigma.0$ fall under the following schemata:

Ax.0. A, where $A \in$ *Taut*,

Ax.1. $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$,

[1]In the original presentation (cf. Wiśniewski 1995), $\Box$ and $\Diamond$ were replaced by S and D, respectively.

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