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Propositions, Possible Worlds, and Recursion*

1. Propositions are sometimes regarded as sets of possible worlds. This idea, the first explicit formulations of which are attributed by Cresswell (1972) to Montague (1969) and Stalnaker (1970), *prima facie* seems quite attractive. However, the *identification* of propositions with sets of possible worlds quickly puts us into a trouble: *there are too many propositions*. For let us suppose that there exist denumerably many (by "denumerable" we mean, here and below, "countably infinite") possible worlds. Assume also that a language in question comprises denumerably many (declarative) sentences. Thus, by Cantor's diagonal argument, the cardinality of the set of propositions is greater than the cardinality of the set of sentences. Now suppose that the assignment of propositions to sentences is univocal, i.e. there is exactly one proposition that corresponds to a sentence. It follows that there are propositions which are not assigned to any sentence – generally speaking, propositions which are not expressed by any sentence. Otherwise we arrive at a contradiction. The situation is analogous when there are more than denumerably many possible worlds. And nothing changes when the language in question is formal and thus the sentences of the language are its well-formed formulas.

2. The above drawback is easily visible.[1] In order to get rid of it one has to take into consideration only an at most denumerable family of sets of

[1]The situation resembles that known from discussions on the problem of adequacy of intensional semantics. If propositions are sets of possible worlds, then, by Cantor's diagonal argument, there are more propositions than possible worlds. On the other hand, by the so-called principle of plenitude (see Lindström 2009), which originates from Kaplan (see e.g. Kaplan 1994; the relevant principle was introduced by him in the sixties) there are at least as many possible worlds as there are propositions. This is the essence of Russell-Kaplan Paradox, widely discussed in the literature.

*First published in *Logic and Logical Philosophy* **20**, 2011, pp. 123–131. Reprinted here with kind permission from the Nicolaus Copernicus University Press.

possible worlds and to identify propositions with elements of the family. Yet it is unclear what (if any) is the right criterion for choosing the family.

3. Another shortcoming of the analysed account of propositions is widely known: it leads to a very coarse individuation of propositions. In particular, there is only one proposition that corresponds to any contradiction, namely the empty set. Similarly, there is only one proposition that corresponds to any logical truth.[2]

4. The main aim of this paper is to point out a certain further difficulty faced by the reduction of propositions to sets of possible worlds. The difficulty arises on the condition that tools and results of (classical) recursion theory are applicable, although in an indirect manner, to sets of possible worlds. This, in turn, presupposes that not only sentences but also possible worlds are represented by natural numbers in a one-to-one way.

The details of the relevant mapping will not play any role in our reasoning: we simply assume that a certain mapping exists, is fixed, and is such that each sentence and each possible world is represented by exactly one natural number, but there is no natural number which represents both a sentence and a possible world. For brevity, we shall call the natural number which represents a possible world or a sentence the code of the possible world or the sentence.

5. Let W be an arbitrary but fixed infinite *recursive* set of possible worlds. Or, to be more precise, let W be a denumerable set of possible worlds such that the set $[W]$ of codes of elements of W is recursive.

Call a proposition *any* subset of W.

Consider a language, $\mathcal{L}$, which comprises denumerably many sentences and for which there exists an assignment of propositions (taken from $\wp(W)$, i.e. the power set of W) to sentences of $\mathcal{L}$. We assume that this assignment is univocal. However, we do not prejudge the nature of the assignment. Traditionally, the proposition assigned to sentence A is conceived as the set of all the possible worlds such that A is true in each world which belongs to the relevant set. We neither assume nor deny that it is the case, however. We simply suppose that to each sentence there corresponds exactly one proposition.

The proposition assigned to sentence A will be referred to as $|A|$. The set of codes of elements of $|A|$ will be designated by $[|A|]$. In general, by $[X]$ we designate the set of codes of all the elements of X.

[2] For a recent discussion see e.g. Berto (2010).

Let us introduce some auxiliary notions.

DEFINITION 1. *Sentence A of $\mathcal{L}$* expresses *proposition $X \in \wp(W)$ iff* $X = |A|$.

DEFINITION 2. *Proposition $X \in \wp(W)$ is:*

1. infinite *iff X is denumerable,*
2. recursive *iff $[X]$ is recursive,*
3. recursively enumerable *iff $[X]$ is recursively enumerable.*

There is no space for defining the concepts of recursion theory used; they are basic and thus we assume that a reader is familiar with them. As usual, we abbreviate "recursively enumerable" as "r.e.".

6. We need one more concept. Let Σ be the set of all the sentences of $\mathcal{L}$. We define the following relation $R^* \subseteq \Sigma \times W$ between sentences and possible worlds:

$$(\forall A \in \Sigma)(\forall w \in W)(R^*(A, w) \leftrightarrow w \in |A|).$$

Thus $R^*(A, w)$ holds just in case world w belongs to the proposition expressed by A. Since sentences and possible worlds are, by assumption, uniformly coded by natural numbers, there exists a 1–1 function, say, g, such that $g(x)$ is the sentence/possible world coded by x. Moreover, there exists exactly one relation $\widehat{R} \subseteq [\Sigma] \times [W]$ between codes of sentences and codes of possible worlds such that the following holds:

$$(\forall x \in [\Sigma])(\forall y \in [W])(\widehat{R}(x, y) \leftrightarrow R^*(g(x), g(y))).$$

We say that the assignment of propositions to sentences is *effective* iff $\widehat{R}$ is an r.e. relation. The underlying idea is: if a world is an element of the proposition expressed by a sentence, this can be effectively established.

7. Now let us ask:

($\star$) Is it possible that each recursive proposition $X \in \wp(W)$ is expressed by some sentence of $\mathcal{L}$?

A remark is in order. There exist denumerably many infinite recursive subsets of an infinite recursive set. Thus, to avoid triviality, let us suppose that the language $\mathcal{L}$ involves denumerably many sentences each of which expresses an infinite proposition. Otherwise the answer to ($\star$) would be negative from the very beginning, for the assignment of propositions to sentences is supposed to be univocal.

8. Assume that $\mathcal{L}$ is a language in which the assignment of propositions to sentences is effective. Let Ψ be the set of all the sentences of $\mathcal{L}$ that express infinite propositions, i.e.:

(1) for each $A \in \Psi$: $|A|$ is denumerable.

Suppose that:

(2) Ψ is denumerable,

(3) $[\Psi]$ is r.e.

One can prove that then *there exists an infinite family Ξ of infinite recursive subsets of $[W]$ such that for each $Y \in \Xi$ and each $A \in \Psi$*:

$$Y \neq [|A|].$$

Now let us take an arbitrary but fixed $Y \in \Xi$. Consider $g(Y)$, i.e. the image of Y under the function g that assigns possible worlds as well as sentences to their codes (see Section 6). Clearly, $g(Y)$ is a proposition belonging to $\wp(W)$. On the other hand, the image of $[|A|]$ under g is $|A|$, i.e. the proposition expressed by A, for any sentence A. The above result yields that $g(Y)$ is different from $|A|$, for each $Y \in \Xi$ and each $A \in \Psi$. In other words, *there are recursive infinite propositions*[3] *belonging to $\wp(W)$ which are not expressed by any sentence of $\mathcal{L}$ / are not assigned to any sentence of $\mathcal{L}$*. Thus, if the assumptions specified above hold, the answer to the question $(\star)$ is *negative*.

Observe that the answer to a more general question:

$(\star')$ Is it possible that each recursively enumerable proposition X from $\wp(W)$ is expressed by some sentence of $\mathcal{L}$?

is also *negative* under the assumptions made above. The reason is simple: each recursive proposition is an r.e. proposition as well.

9. For conciseness, let $R^{\rightarrow}x = \{y \in rng(R) : xRy\}$, where $rng(R)$ is the range of a binary relation R.

The formal result presented in Section 8 is obtained in two steps.

(I) We define a certain relation R' by:

$$\forall x \forall y (R'(x, y) \leftrightarrow x \in [\Psi] \wedge \widehat{R}(x, y)).$$

Recall that $[\Psi]$ and $\widehat{R}$ are, by assumption, r.e. Hence R' is r.e. as well. We obtain an effective deeply infinite double frame:[4]

[3] Actually, denumerably many of them.

[4] A *double frame* is an ordered triple $\langle \Phi, \Gamma, R \rangle$, where Φ, Γ are non-empty sets and $R \subseteq \Gamma \times \Phi$ is a relation whose domain is Γ. A double frame $\langle \Phi, \Gamma, R \rangle$ is *deeply infinite* if Φ and Γ are countably infinite sets, and each set $R^{\rightarrow}x$ is infinite, for all $x \in \Gamma$. A double frame $\langle \Phi, \Gamma, R \rangle$ is *effective* if Φ and Γ are sets of natural numbers, Φ is recursive, Γ is r.e. and R is an r.e. relation. Cf. Wiśniewski & Pogonowski (2010).

$$\langle [W], [\Psi], R' \rangle.$$

Note that R' has the following property:

(•) $\{X \subseteq [W] : X = (R')^{\rightarrow}[A] \text{ for some } A \in \Psi\} =$
$$= \{X \subseteq [W] : X = [|A|] \text{ for some } A \in \Psi\}.$$

(II) We make use of the following theorem:[5]

The Recursive Jump Theorem (Wiśniewski & Pogonowski 2010). *For any deeply infinite effective double frame $\langle \Phi, \Gamma, R\rangle$ there exists an infinite family Ξ of infinite recursive subsets of Φ such that each element of Ξ is different from any $R^{\rightarrow}x$, for all $x \in \Gamma$.*

10. Let us now reverse the picture by assuming that the language $\mathcal{L}$ and its semantics are built in such a way that each recursive proposition $X \in \wp(W)$ is expressed by some sentence of $\mathcal{L}$ and the relevant assignment (of propositions to sentences) is still univocal. It follows that the set Ψ of all the sentences of $\mathcal{L}$ which express infinite propositions is denumerable.

Now, by the result presented in Section 8, at least one of the following, (**A**) or (**B**), holds:

(**A**) $[\Psi]$ is not r.e.

An r.e. set is the set of values of a partial recursive function, and partial recursive functions correspond to algorithms. Since sentences of $\mathcal{L}$ are coded by natural numbers, it follows that the set Ψ is not positively decidable. In other words, there is no algorithmic procedure which is capable to identify, in a finite number of steps, each element of the set of sentences expressing infinite propositions.

(**B**) the assignment of propositions to sentences is not effective.

Strictly speaking, (**B**) means that the relation $\widehat{R}$ which fulfils the condition:

$$(\forall x \in [\Sigma])(\forall y \in [W])(\widehat{R}(x,y) \leftrightarrow R^*(g(x), g(y))).$$

is not r.e. But given that R^* is defined by:

$$(\forall A \in \Sigma)(\forall w \in W)(R^*(A,w) \leftrightarrow w \in |A|).$$

and g is the "decoding" function, that is, a function that recovers sentences and possible worlds from their codes, it follows that there occurs at least one sentence such that some world(s) belong(s) to the proposition expressed

[5]For a direct application of the theorem in the area we are interested in here see Wiśniewski & Pogonowski (2010), pp. 38–39. The setting adopted in the present paper is more general.

by the sentence, but this can not be effectively established. Thus each algorithmic procedure whose outputs are true statements saying that a world belongs to the proposition expressed by the sentence, is incomplete in the sense that it does not "reach", in a finite number of steps, a certain true statement of this kind.

11. The results of this paper can be interpreted in two ways. First, as "deep" philosophical claims, telling something about the necessity of existence of inexpressible propositions. Second, as a, just another one, argument against the identification of propositions with sets of possible worlds. The reader is free to choose between these options.

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