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REDUCIBILITY OF SAFE QUESTIONS TO SETS OF ATOMIC YES-NO QUESTIONS

1. Yes-No Questions

By a yes-no question we mean a question which can be satisfactorily answered by saying either "yes" or "no." Usually the answer "yes" is regarded as an abbreviation of the sentence which can be obtained from the question by the conversion from the interrogative to the declarative mood, whereas the answer "no" is interpreted as an abbreviation of the negation of this sentence. According to the analysis which can be found in most textbooks, negation should be understood here in the sense of classical logic. For example, the interrogative sentence:

(1) Did John marry Joan?

is interpreted along the lines sketched above as expressing the question which has the following possible and just-sufficient (i.e. direct) answers:

(2) John married Joan.
(3) It is not the case that John married Joan.

These answers contradict each other; the question must have a true direct answer and thus is a safe question. Let us call a question whose set of possible and just-sufficient answers consists of a sentence and its (classical) negation a *simple yes-no question*.

It can hardly be said, however, that each natural-language yes-no question can be adequately analyzed as a simple yes-no question. Even the interrogative sentence (1) can be interpreted differently. Let us observe that (1) can be pronounced with the following intonations (among others):

(4) *Did* John marry Joan?
(5) Did *John* marry Joan?

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(6) Did John marry *Joan*?

In all of the above cases the answer "yes" is an abbreviation of (2). The meaning of the reply "no," however, changes from case to case. As far as (4) is concerned, the meaning of "no" is expressed by the sentence (3). Yet, in the case of (5) "no" means:

(7) Someone else (not John) married Joan.

The sentences (2) and (7) share the same background assumption, namely that someone married Joan; what is questioned is whether it was John. Similarly, in the case of (6) the reply "no" means:

(8) John married someone else (not Joan).

Both (2) and (8) presuppose that John married someone; what is inquired via (6) is whether it was Joan. So the questions (5) and (6) interpreted along the lines sketched above are no longer safe questions: it may happen that no direct answer to them is true (John may be a monk, and Joan a declared feminist). The questions (5) and (6) construed in the above manner are examples of *focussed yes-no questions*.

Let's now consider the following interrogative sentence:

(9) Did John marry Joan and love her?

At first sight it may look like (9) expresses a yes-no question. Clearly the reply "yes" means:

(10) John married Joan and loved her.

But what is the meaning of the reply "no"? The negation of (10), viz.:

(11) It is not the case that John married Joan and loved her.

is a possible answer, but usually not a just-sufficient one. The just-sufficiency condition is fulfilled by any of the following:

(12) John married Joan, but he didn't love her.
(13) John didn't marry Joan, but he loved her.
(14) John neither married Joan nor loved her.

Taking this interpretation for granted (of course, it is not the only possible one), (9) expresses a *conjunctive question* and not a yes-no question: it cannot be satisfactorily answered by saying either "yes" or "no." Yet, its grammatical form is similar to that of yes-no questions. And it is still a safe question: it must have a true direct answer.

Let's now consider the famous spouse-beating question:

(15) Has Joan stopped beating her husband?

It is well-known that the reply "yes" should be construed as something like:

(16) Joan has beaten her husband but has stopped.

whereas the reply "no" means:

(17) Joan has beaten her husband and has not stopped.

If Joan never beat her husband, both (16) and (17) are false. So (15) is not a safe question. It is also clear that (15) is not a simple yes-no question: it is a *conditional yes-no* question. It is worth emphasizing that not only questions about forbearance can be regarded as conditional yes-no questions. Let's consider:

(18) Given that Russia will constantly oppose NATO's enlargement, will Ukraine join NATO?

The reply "yes" to (18) means:

(19) Russia will constantly oppose NATO's enlargement but Ukraine will join NATO.

The reply "no" to (18) means, in turn:

(20) Russia will constantly oppose NATO's enlargement and Ukraine will not join NATO.

The above considerations show that it is not the case that each yes-no question of a natural language should be construed as a simple yes-no question and thus a safe question. Of course, this fact is known to many logicians and linguists.[1] It is also completely obvious that most questions cannot be construed as yes-no questions of any type. But the simple yes-no questions are customarily regarded as the *epistemologically prior* questions. Among simple yes-no questions, in turn, questions whose sets of direct answers consist of an atomic sentence and its (classical) negation seem to have the *logical priority*; we shall call them *atomic yes-no questions*. So the following questions arise: is it possible to reduce any question to simple yes-no questions? If not, what questions can be reduced in this way? And is it the case that each question that can be

[1] See, e.g., Koj (1972), Belnap (1969), Hajièová (1983), Kiefer (1980, 1988). This is not to say, however, that all of them would accept the analysis of the above questions presented here. For details of this analysis, see Wiśniewski (1995), Chapter 3.

reduced to simple yes-no questions can be also reduced to atomic yes-no questions?

2. Reducibility

As usual, the answer depends on the meaning of the crucial term. Let us observe that one can speak of: (a) reducibility of a (single) question of some kind to a (single) question of another kind, or (b) reducibility of a (single) question of some kind to a set of questions of some kind or kinds. By and large, the former concept has been clarified in two ways: as an equivalence within a given calculus (see e.g. Åqvist 1965) or as some equivalence relation between questions which is defined in terms of (set-theoretic or semantic) relations between sets of their direct answers or in terms of relations between sets of presuppositions (cf. mainly Kubiński 1980, but also Belnap and Steel 1976). It is not surprising that no general results have been obtained in this perspective: it would have been rather strange if one had proved that, for example, a which-question can be reduced, in any reasonable meaning of the word "reduction," to a simple yes-no question. The situation is different, however, in the case of the latter concept of reducibility. Now reducibility of a (initial) question to a set of (auxiliary) questions is under consideration. In order to find the correct answer to an initial question we usually pass to a number of auxiliary questions and try to answer them. This can be done in many ways; yet, there are some underlying logical relations which enable us to do this in a (relatively) safe and efficient way. In Wiśniewski (1994), the concept "a question is reducible to a non-empty set of questions" is defined in semantic terms. The proposed definition pertains to a formalized language: it is a first-order language enriched with questions and supplemented with a model-theoretic semantics. Some conditions are imposed on the language under consideration. In particular, it is assumed that to each question of the language there is assigned a set of declarative sentences of the language which contains at least two sentences; elements of this set are called *direct answers* to the question. Direct answers are defined syntactically; on the other hand they are regarded as the possible and just-sufficient (providing neither less nor more information than is called for) answers. They may be true or false. The intuitions which underlie the proposed definition of reducibility can be briefly described as follows. First, an initial question and the questions to which the initial question is reducible must be *mutually sound*: it is required that if a question Q is reducible to a set of questions Φ, then Q has a true direct answer if and only if each question in Φ has a true direct answer. The second requirement is the *efficacy condition*: it is required that an initial question can always be answered by answering the questions to which it is reducible. To be more precise, it is required that if a question Q is reducible to a set of questions Φ, then each set made up of direct answers to the questions of Φ which contains exactly one direct answer to each question of Φ must entail some direct answer(s) to Q. The last requirement is the *relative simplicity condition*: all questions to which a given question is reducible are supposed to be no more complex than the initial question in the sense that no one of those questions has more direct answers than the initial question. The relevant concept of reducibility is then defined in semantic terms: we will introduce the definition below. Yet, in order to give a simple example let us observe that the question:

(9) Did John marry Joan and love her?

interpreted as a conjunctive question is reducible *int.al.* to the set of questions whose elements are:

(21) Did John marry Joan?
(22) Did John love Joan?

provided that these are construed as simple yes-no questions.

When the concept of reducibility is clarified in the above manner, some solutions to our main problem emerge. First, it may be proved that each safe question (i.e., roughly, a question which must have a true direct answer; see below) is reducible to some set of questions exclusively made up of simple yes-no questions. It can also be proved that the relevant set of simple yes-no questions is finite if the initial question has a finite number of direct answers or entailment in the language is compact. We may even go further in this direction: one may prove that safety amounts to reducibility to sets of simple yes-no questions. Yet, in the case of risky questions, that is, questions which are not safe, the situation is far more complicated. It can be proved that each risky but proper and regular question[2] is reducible to some set of questions made up of simple yes-no question(s) and exactly one question which is not a simple yes-no question, but nevertheless has exactly two direct answers.

[2] Roughly, a question Q is proper and regular if no direct answer to it is entailed by the set of presuppositions of Q, but nevertheless the question Q has a presupposition whose truth guarantees the existence of a true direct answer to it (that is, which multiple-conlusion entails the set of direct answers to Q). By a presupposition of a question Q we mean here a d-wff which is entailed by each direct answer to Q.

The negative result, however, is that no risky question is reducible to a set of questions whose elements are only simple yes-no questions. (For proofs and further results, see Wiśniewski 1994; see also Wiśniewski 1995, Chapter 7.)

Risky questions are thus not reducible to homogenous sets of simple yes-no questions. But safe questions are reducible that way. So we may ask: is it the case that each safe question is reducible to some set of logically prior simple yes-no questions, that is, atomic yes-no questions? As we will see, the answer is rather complicated: there are conditions under which it is the case and conditions in which it is not.

In order to go on we have to introduce some logical apparatus.

3. The Logical Basis

First, we need some formalized language whose meaningful expressions are either declaratives or questions. There are many methods of constructing such languages (for an overview see, for example, Harrah 2002, or Wiśniewski 1995). Most of the details of the construction are irrelevant for the purposes of this analysis, however. Thus we only assume that we have at out disposal some formalized language $\mathscr{L}$ which consists of a declarative part as well as of an erotetic part and fulfills certain conditions. The declarative part of $\mathscr{L}$ is a first-order language with identity whose vocabulary contains all the connectives $\neg$, $\vee$, &, $\supset$, $\equiv$, the quantifiers $\forall$, $\exists$, and at least one closed term (i.e., a term with no individual variable). As far as the declarative part of $\mathscr{L}$ is concerned, the concepts of term, atomic well-formed formula, (declarative) well-formed formula (d-wff for short), freedom and bondage of variables, etc., are defined as usual; by a sentence of $\mathscr{L}$ we mean a d-wff of $\mathscr{L}$ without free variables and by a sentential function we mean a d-wff which is not a sentence. The vocabulary of the erotetic part of $\mathscr{L}$ contains some expressions which enable us to form questions of this language. Questions of $\mathscr{L}$ are not d-wffs but they are the meaningful expressions of the erotetic part of $\mathscr{L}$. We do not decide, however, what is the particular form of questions of $\mathscr{L}$: they may be constructed in some way or another (see, *inter alia*, Belnap and Steel 1976; Kubiński 1980; Harrah 2002; or Wiśniewski 1995). Yet, we assume that the following conditions are met: (1) the syntax of $\mathscr{L}$ assigns to each question of $\mathscr{L}$ an at least two-element set of sentences of $\mathscr{L}$; these sentences are called *direct answers* to the question and, looking from the pragmatic point of view, are regarded as the possible and just-sufficient answers; (2) for each sentence A of $\mathscr{L}$

there exists the corresponding simple yes-no question of $\mathscr{L}$ whose set of direct answers consists of the sentence A (called the *affirmative direct answer*) and the sentence $\neg A$ (called the *negative direct answer*); (3) there are questions of $\mathscr{L}$ which are not simple yes-no questions; (4) the set of questions of $\mathscr{L}$ is denumerable.[3]

Some comments on the condition (1) are in order here. We require the question-answer relationship in $\mathscr{L}$ to be purely syntactical. Moreover, we claim that it is the logical form of a question of $\mathscr{L}$ which determines what counts as not only possible but also just-sufficient answer to the question. It is obvious that things look differently in natural languages. Although the syntactical form of a question is usually an important factor, it is not always the decisive one. There are cases in which some meaning components of a natural-language question play an important role. And in general, as many linguists and philosophers pointed out, there are cases in which it is strongly context-dependent what sentences may be counted as the possible and just-sufficient answers to some question: such pragmatic factors as, for example, intonation or other focus indicators, or the position of a question in a text or in a utterance, or the state of knowledge of the questioner, or his/her intentions, or the speech situation are relevant. Yet, questions of $\mathscr{L}$ are questions of a formalized language and they only *represent* natural-language questions. The relation of representation we have in mind can be briefly described as follows: a question Q of $\mathscr{L}$ represents a question Q^* of a natural language *construed in such a way* that the possible and just-sufficient answers to Q^* have the *logical form* of direct answers to Q. Thus we do not say that there is one-to-one correspondence between natural-language interrogative sentences and questions of $\mathscr{L}$: if a natural-language question admits many readings, it has many representations. The richer the interrogative part of $\mathscr{L}$ is, the more natural-language questions and/or their admissible readings can be represented. However, beyond the scope of such a representation system (and thus also beyond the scope of our analysis) are the so-called *open* natural-language questions, that is, questions whose possible and just-sufficient answers cannot be defined even if all the relevant syntactical/semantical/ pragmatical factors are known.

The condition (1), however, not only requires that the question-answer relationship in $\mathscr{L}$ is purely syntactical, but also claims, first, that each question of $\mathscr{L}$ has at least two direct answers and, second, that each

[3] In Wiśniewski (1994), some additional condition are imposed on $\mathscr{L}$ as well. Yet, we will not make use of them in this paper.

direct answer is a sentence. The motivation for the first claim is philosophical: we think that a necessary condition of being a question is to present at least two "alternatives" or conceptual possibilities among which some selection can be made. The "Hobson choice" questions are thus excluded, but rhetorical questions are allowed – the selection need not be rational.[4] Concerning the second clause: we want the direct answers to be the just-sufficient answers and those can be expressed only by sentences.

We shall use the letters $A, B, C, \ldots$ (with subscripts if needed) as metalinguistic variables for d-wffs of $\mathcal{L}$, and the letters X, Y, Z as metalinguistic variables for sets of d-wffs of $\mathcal{L}$. The symbols $Q, Q_1, \ldots$ will be used as metalinguistic variables for questions and the capital Greek letters (with or without subscripts) as metalinguistic variables for sets of questions. The set of direct answers to a question Q will be referred to as dQ. In the metalanguage of $\mathcal{L}$ we assume the Bernays-von Neumann-Gödel version of set theory; we adopt here the standard set-theoretical terminology and notation. Sometimes we shall write "iff" instead of "if and only if."

The semantics of $\mathcal{L}$ is basically the model-theoretic one. By an interpretation of $\mathcal{L}$ we mean an ordered pair $<U, f>$, where U is a non-empty set (the *universe*) and f is the *interpretation function* defined on the set of non-logical and non-erotetic constants of $\mathcal{L}$ (that is, predicates, individual constants and – if there are any – function symbols) in the standard way. Of course, there are many interpretations of $\mathcal{L}$. If $\mathcal{I}$ is an interpretation, then by a *$\mathcal{I}$-valuation* we mean an infinite sequence of the elements of the universe of $\mathcal{I}$. The concepts of *value* of a term under a $\mathcal{I}$-valuation and of *satisfaction* of a d-wff in an interpretation $\mathcal{I}$ by a $\mathcal{I}$-valuation are defined in the standard manner. A d-wff A is said to be *true* in an interpretation $\mathcal{I}$ if and only if A is satisfied in $\mathcal{I}$ by all $\mathcal{I}$-valuations; by a *model* of a set of d-wffs we mean an interpretation in which all the d-wffs of this set are true. Note that the concept of truth does not apply to questions of $\mathcal{L}$. In the case of questions, however, we use the concept of soundness. A question Q is said to be *sound* in an interpretation $\mathcal{I}$ if and only if at least one direct answer to Q is true in $\mathcal{I}$.

The further semantical concepts pertaining to $\mathcal{L}$ are defined by means of the concept of *normal interpretation* of $\mathcal{L}$. Yet, the language $\mathcal{L}$ was characterized only in a schematic manner and in fact there are many languages which fulfil the conditions specified so far. For that reason we

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[4] Some logical theories of questions (for example, Belnap's theory or Kubiński's theory) allow questions of formalized languages which have only one direct answer, but it seems that this step is motivated rather by the pursuit of generality than other reasons.

only assume that the class of interpretations of $\mathcal{L}$ includes a non-empty subclass (not necessarily a proper subclass) of normal interpretations, but we do not decide what the normal interpretations are in each particular case. If the declarative part of $\mathcal{L}$ is an applied first-order language, normal interpretations can be defined as those in which some meaning postulates and/or axioms are true. Normal interpretations can also be defined for purely erotetic reasons. If $\mathcal{L}$ contains questions about objects satisfying some conditions, it would be natural to define normal interpretations as those in which all the objects called for have names (are values of some closed term(s)): by doing so we would avoid the paradoxical consequence that there are objects which satisfy the appropriate conditions, but nevertheless the corresponding questions have no true answers. There are also other possibilities of defining normal interpretations (for more information, see Wiśniewski 1995, pp. 104-105). We do not even exclude that the class of normal interpretations of some language of the considered kind is equal to the class of all interpretations of this language. Yet, for the purposes of this analysis the assumption about the existence of a non-empty class of normal interpretations is sufficient.

By means of normal interpretations we shall define the relevant concepts of entailment in $\mathcal{L}$. We will introduce two concepts of entailment: multiple-conclusion entailment being a relation between sets of d-wffs and (single-conclusion) entailment understood as a relation between sets of d-wffs and single d-wffs.

> DEFINITION 1. A set of d-wffs X of $\mathcal{L}$ *multiple-conclusion entails* (mc-entails for short) *in* $\mathcal{L}$ a set of d-wffs Y of $\mathcal{L}$ iff the following condition holds:
>
> > (#) for each normal interpretation $\mathcal{I}$ of $\mathcal{L}$: if all the d-wffs in X are true in $\mathcal{I}$, then at least one d-wff in Y is true in $\mathcal{I}$.
>
> DEFINITION 2. A set of d-wffs X of $\mathcal{L}$ *entails in* $\mathcal{L}$ a d-wff A of $\mathcal{L}$ iff A is true in each normal interpretation of $\mathcal{L}$ in which all the d-wffs in X are true.

Note that the above concepts are defined in terms of truth and not of satisfaction. Note also that in the general case mc-entailment cannot be defined in terms of (single-conclusion) entailment. For instance, assume that the declarative part of $\mathcal{L}$ is the language of Classical Predicate Calculus and that each interpretation of $\mathcal{L}$ is a normal one. Then the singleton set $\{A \vee B\}$, where A, B are atomic sentences, mc-entails the set $\{A, B\}$, but neither A nor B is entailed by the set $\{A \vee B\}$. On the other hand, entailment can be defined in terms of mc-entailment as multiple-conclusion entailment of a singleton set.

The concept of multiple-conclusion entailment proved it usefulness in the logic of questions in many ways. (For the properties of mc-entailment, see Shoesmith & Smiley 1978; see also Wiśniewski 1995, pp. 107-113.)

Since the concept of normal interpretation was left unspecified, the same pertains to the above concepts of entailment. But since the class of normal interpretations was assumed to be a subclass of the class of all interpretations, logical entailment (defined in the manner similar to that of Definition 2, but with respect to any interpretation) is a special case of entailment in $\mathscr{L}$. We may also say that, in particular, any disjunction of sentences (or, to be more precise, a singleton set containing this disjunction) mc-entails in $\mathscr{L}$ the set made up of the appropriate disjuncts.

In what follows the specification "in $\mathscr{L}$" will normally be omitted. We shall use the symbol $\Vdash$ for mc-entailment in $\mathscr{L}$ and the symbol $\models$ for entailment in $\mathscr{L}$. We shall write $A \Vdash Y$ instead of $\{A\} \Vdash Y$.

The relation $\Vdash$ is said to be compact if and only if for any sets of d wffs X, Y such that $X \Vdash Y$ there exist a finite subset X_1 of X and a finite subset Y_1 of Y such that $X_1 \Vdash Y_1$. In the case of $\models$ the concept of compactness is understood in the standard way. It may be proved that mc-entailment in a language is compact if and only if entailment in this language is compact. However, we neither claim nor deny that entailment in $\mathscr{L}$ and mc-entailment in $\mathscr{L}$ are compact. Compactness of entailment in a language depends on the conditions imposed on the class of normal interpretations of the language and there are languages of the considered kind in which entailment is compact and languages in which it is not.

We are now ready to define the concept of reducibility of questions.

DEFINITION 3. A question Q is *reducible to* a non-empty set of questions Φ iff

(i) for each direct answer A to Q, for each question Q_i of Φ: A mc-entails the set of direct answers to Q_i, and
(ii) each set made up of direct answers to the questions of Φ which contains exactly one direct answer to each question of Φ entails some direct answer to Q, and
(iii) no question in Φ has more direct answers than Q.

For conciseness, the non-emptiness clause will be omitted in the sequel. Also for the sake of brevity we shall introduce the notion of a $\mu(\Phi)$-set. Let Φ be a non-empty set of questions. By a $\mu(\Phi)$-*set* we mean a set made up of direct answers to the questions of Φ which contains exactly one direct answer to each question of Φ. By means of this concept the second clause of Definition 3 can be expressed as follows: each $\mu(\Phi)$-set entails some direct answer to Q. When saying that no question in Φ has more direct answer than Q we mean that the cardinality of the set of direct answers to any question of Φ is not greater than the cardinality of the set of direct answers to Q.

Let us finally clarify the erotetic concepts of safety and riskiness. A question Q of $\mathcal{L}$ is said to be *safe* if and only if Q is sound (has a true direct answer) in each normal interpretation of $\mathcal{L}$; otherwise Q is said to be *risky*. (It is easy to observe that safety can be also defined in terms of mc-entailment: a question Q is safe iff the set of direct answers to Q is mc-entailed by the empty set.) Note that a question can be safe although no direct answer to it is valid (i.e. is true in each normal interpretation of the language)! Of course, each simple yes-no question is safe, but there are also safe questions which are not simple yes-no questions.

In what follows we will be frequently speaking of simple yes-no questions, so, to simplify matters, we need a temporary notation for them. We shall write them down as $?\ \{A, \neg A\}$. Under this notational convention the signs ?, {, } belong to the (erotetic part of the) object-language[5]. Yet, we might have adopted some other notational convention for simple yes-no questions as well. The advantage of this one is that it makes explicit what the direct answers to a simple yes-no question are: these are the sentences enclosed in { }.

Let us finally recall that an *atomic yes-no question* is a simple yes-no question whose affirmative direct answer is an atomic sentence (i.e., a sentence built up of a predicate and closed term(s)) and whose negative direct answer is the negation of this atomic sentence. In other words, an atomic yes-no question has the form $?\ \{B, \neg B\}$, where B is an atomic sentence.

4. The Quantifier-Free Case

It can be shown that in the case of quantifier-free safe questions the reduction to homogenous sets of atomic yes-no questions is always possible. By a quantifier-free question we mean a question whose direct answers contain no occurrence of a quantifier.

Let us prove

[5] Of course, the brackets { } also occur in the metalanguage in their normal roles.

LEMMA 1: *Let A be a quantifier-free sentence. Then the simple yes-no question* ? $\{A, \neg A\}$ *is reducible to some finite set of atomic yes-no questions.*

PROOF: Let us observe that the clauses (i) and (iii) of the definition of reducibility are fulfilled by each set made up of atomic yes-no questions with respect to any "initial" question. The clause (i) is fulfilled because the set of direct answers to an atomic yes-no question is mc-entailed by any d-wff; the clause (iii) is satisfied since each question has at least two direct answers and an atomic yes-no question has exactly two direct answers. So it remains to be shown that for each quantifier-free sentence A there exists a finite set of atomic yes-no questions Φ such that for each $\mu(\Phi)$-set Y, Y entails the sentence A or the sentence $\neg A$.

The proof goes on by induction on the structure of A.

(1) Assume that A is an atomic sentence. Let $Q = ?\ \{A, \neg A\}$. So $\{Q\}$ is a finite set made up of atomic yes no-questions. On the other hand, it is obvious that $\{A\} \models A$ and $\{\neg A\} \models \neg A$.

(2) Assume that A is of the form $\neg B$. By induction hypothesis there exists a finite set Φ of atomic yes-no questions such that for each $\mu(\Phi)$-set Y we have $Y \models B$ or $Y \models \neg B$. But $B \models \neg A$ and $\neg B \models A$. So each $\mu(\Phi)$-set entails A or entails $\neg A$.

(3) Assume that A is of the form $B \,\&\, C$. By induction hypothesis there are a finite set Φ_1 of atomic yes-no questions and a finite set Φ_2 of atomic yes-no questions such that for each $\mu(\Phi_1)$-set Y we have $Y \models B$ or $Y \models \neg B$, and for each $\mu(\Phi_2)$-set Z we have $Z \models C$ or $Z \models \neg C$. On the other hand, the following hold:

(a) $\{B, C\} \models A$,
(b) $\{B, \neg C\} \models \neg A$,
(c) $\{\neg B, C\} \models \neg A$,
(d) $\{\neg B, \neg C\} \models \neg A$.

Let $\Phi = \Phi_1 \cup \Phi_2$. Each $\mu(\Phi)$-set equals to a union of a $\mu(\Phi_1)$-set and a $\mu(\Phi_2)$-set. But each $\mu(\Phi_1)$-set entails B or entails $\neg B$, and each $\mu(\Phi_2)$-set entails C or entails $\neg C$. So by the conditions (a) – (d) each $\mu(\Phi)$-set entails A or entails $\neg A$. It is obvious that Φ is a finite set made up of atomic yes-no questions.

(4) Assume that A is of the form $A \vee B$. We proceed as above by means of the following facts:

(e) $\{B, C\} \models A$,
(f) $\{B, \neg C\} \models A$,

(g) $\{\neg B, C\} \models A$,
(h) $\{\neg B, \neg C\} \models \neg A$.

(5) Assume that A is of the form $B \supset C$. We proceed analogously as above by using the following:

(i) $\{B, C\} \models A$,
(j) $\{B, \neg C\} \models \neg A$,
(k) $\{\neg B, C\} \models A$,
(l) $\{\neg B, \neg C\} \models A$.

(6) Assume finally that A is of the form $B \equiv C$. We use the following:

(m) $\{B, C\} \models A$,
(n) $\{B, \neg C\} \models \neg A$,
(o) $\{\neg B, C\} \models \neg A$,
(p) $\{\neg B, \neg C\} \models A$.

■

Lemma 1 yields

THEOREM 1. *Each quantifier-free simple yes-no question is reducible to some finite set of atomic yes-no questions.*

Thus quantifier-free simple yes-no questions are reducible to sets of logically prior questions, that is, atomic yes-no questions.

Let us now consider the possibility of reduction of any quantifier-free safe question to a homogenous set of atomic yes-no questions.

We shall first prove

THEOREM 2. *If Q is a quantifier-free safe question, then Q is reducible to some set of quantifier-free simple yes-no questions; if moreover Q has a finite number of direct answers or entailment in the language is compact, then Q is reducible to some finite set of quantifier-free simple yes-no questions.*

PROOF: Let Q be a quantifier-free safe question. Direct answers are sentences and the set of direct answers to each question is at most countable. Let $\mathbf{s} = A_1, A_2, \ldots$ be a fixed sequence without repetitions of direct answers to Q such that each direct answer to Q is an element of $\mathbf{s}$. Let us then define the following set of simple yes-no questions:

$$\Phi = \{Q^*: Q^* \text{ is of the form } ?\ \{A_i, \neg A_i\}, \text{ where } i > 1\}$$

In other words, Φ consists of the simple yes-no questions based on the elements of the sequence $\mathbf{s}$ with the exception of the simple yes-no question based on the first element of **s**. It is clear that the clauses (i) and (iii) of Definition 3 are fulfilled by Φ with respect to Q. Let Y be a $\mu(\Phi)$-set. There are two possibilities: (a) the set Y contains some affirmative direct answer(s) to the questions of Φ, (b) the set Y is made up of the negative direct answers to the questions of Φ. If the possibility (a) holds, then – since the affirmative direct answers to the questions of Φ are also direct answers to Q – the $\mu(\Phi)$-set Y entails some direct answer(s) to Q. Suppose that the possibility (b) takes place. Since Q is a safe question, then $\emptyset \Vdash \mathrm{d}Q$. It follows that $Y \models A_1$. So there is a direct answer to Q which is entailed by Y. But Y was an arbitrary $\mu(\Phi)$-set. Therefore Q is reducible to Φ. It is obvious that the set Φ constructed in the above manner consists of quantifier-free simple yes-no questions. Moreover, it is also clearly visible that if Q has a finite number of direct answers, then – since each question has at least two direct answers – the set Φ constructed according to the above pattern is finite and nonempty.

Let us now assume that entailment in $\mathscr{L}$ is compact and that Q is an arbitrary but fixed quantifier-free safe question. If entailment is compact, so is mc-entailment. So there is an at least two-element subset Z of the set of direct answers to Q such that $\emptyset \Vdash Z$ (if $\emptyset$ entails some direct answer to Q, it also mc-entails each at least two-element subset of dQ which contains this answer; if $\emptyset$ does not entail any single direct answer to Q, then by compactness there is an at least two-element finite subset of dQ which is mc-entailed by $\emptyset$). We fix some at least two-element finite subset of the set of direct answers to Q which is mc-entailed by the empty set and then proceed as above; as the outcome we obtain a finite set of quantifier-free simple yes-no questions such that Q is reducible to this set.

■

We can now prove

THEOREM 3. *Each quantifier-free safe question Q is reducible to some set of atomic yes-no questions; if moreover Q has a finite number of direct answers or entailment in the language is compact, then Q is reducible to a finite set of atomic yes-no questions.*

PROOF: Assume that Q is a quantifier-free safe question. According to Theorem 2, Q is reducible to some set of quantifier-free simple yes-no questions (a finite set if Q has a finite number of direct answers or entailment in $\mathscr{L}$ is compact). Let Φ be a fixed set of quantifier-free simple yes-no questions such that Q is reducible to Φ; if Q has a finite number of direct answers or entailment in the language is compact, Φ is supposed to be a finite set. By Theorem 1 each question in Φ is reducible to some finite set of atomic yes-no questions. We associate each question in Φ with exactly one finite set of atomic yes-no questions such that the considered question of Φ is reducible to this set. Let Ψ be the union of sets of atomic yes-no questions associated with the questions of Φ in the above manner. The set Ψ is a homogenous set of atomic yes-no questions. It is clear that the clauses (i) and (iii) of Definition 3 are met by Ψ with respect to *Q*.

Let *Y* be a μ(Ψ)-set. It is easily seen that *Y* entails some direct answer to each question of Φ. So there exists a μ(Φ)-set, say, *X*, such that each normal interpretation which is a model of *Y* is also a model of *X*. But since *Q* is reducible to Φ, the μ(Φ)-set *X* entails some direct answer to *Q*. Therefore the μ(Ψ)-set *Y* entails some direct answer to *Q*. But since *Y* was an arbitrary μ(Ψ)-set, it follows that the clause (ii) of Definition 3 is also fulfilled by Ψ with respect to *Q*. Therefore *Q* is reducible to Ψ, where Ψ is a set of atomic yes-no questions. It is clear that if *Q* has a finite number of direct answers or entailment in the language is compact, then Ψ is a finite set.

■

Theorem 3 shows that in the case of quantifier-free safe questions the reduction to sets of atomic yes-no questions is *always* possible; moreover, it shows that in some cases the reduction to *finite* sets of atomic yes-no questions is possible as well. Note that no assumptions concerning the particular form of the semantics of $\mathscr{L}$ have been used in the proofs of the above theorems; it follows that the reducibility of quantifier-free safe questions to sets of atomic yes-no questions takes place in every language of the considered kind.

5. The General Case

So far we have restricted ourselves to quantifier-free safe questions. But what happens if the initial question is not quantifier-free?

Sometimes the initial safe question *Q* is not quantifier-free, but nevertheless it is reducible to some set of quantifier-free simple yes-no questions. One can easily prove

> THEOREM 4. *Let Q be a safe question. If Q is reducible to some set of quantifier-free simple yes-no questions, then Q is reducible to some set of atomic yes-no questions; if moreover Q is reducible*

to some finite set of quantifier-free simple yes-no questions, then Q is reducible to some finite set of atomic yes-no questions.

PROOF: Similar to that of Theorem 3. It is obvious that if Q is reducible to a finite set of quantifier-free simple yes-no questions, then the resultant set Ψ is finite.

■

Yet, there is no guarantee that each safe question is reducible to a set of quantifier-free simple yes-no questions (although, as we shall see, there is a guarantee that each safe question is reducible to some set of simple yes-no questions). Moreover, there are examples which show that Theorem 3 cannot be generalized to any safe question with respect to any language of the considered kind. Here is a very simple example: assume that the declarative part of $\mathscr{L}$ is the language of Monadic Classical Predicate Calculus and that each interpretation of $\mathscr{L}$ is a normal one. Let us then consider a simple yes-no question of the form $?\ \{\forall P(x), \neg\forall P(x)\}$, where P is a predicate. At first sight it may look as if the above question is reducible to the set made up of atomic yes-no questions of the form $?\ \{P(t), \neg P(t)\}$, where t is a closed term. Yet, the set which contains only the affirmative direct answers to the above questions does not entail any direct answer to the initial question. The reason is that there may exist some “unnamed” elements of the domain which do not satisfy the sentential function $P(x)$.

The above example not only shows that the reducibility to sets of atomic yes-no questions does not always hold, but also suggests a certain sufficient condition whose satisfaction enables reducibility of any safe question to a set of atomic yes-no questions.

Let Ax be a sentential function with exactly one free variable. Let us designate by $\mathbf{S}(Ax)$ the set of sentences which result from the sentential function Ax by proper substitution of a closed term for the variable which occurs free in Ax (i.e. the set of sentences which have the form $A(x/t)$, where t is a closed term). Let us now consider the following condition:

(ω) *for each sentential function Ax with exactly one free variable*, $\exists x\, Ax \;||\!\!=\; \boldsymbol{S}(Ax)$.

The condition (ω) says that for each sentential function with exactly one free variable, the existential generalization of this sentential function multiple-conclusion entails the set of sentences which are instantiations of this sentential function.[6] In other words, the condition (ω) requires that for each *normal* interpretation of the language the truth of the existential generalization of a sentential function with exactly one free variable guarantees that at least one sentence which results from this sentential function by proper substitution of a closed term for the free variable is true. It follows that the set of normal interpretations of the considered language must be a proper subclass of the class of all interpretations of it: as a matter of fact the normal interpretations are those, in which all the elements of the universe have names (to be more precise, the condition (ω) is fulfilled if for each element y of the universe there exists a closed term t such that for any valuation $\mathbf{s}$, y is the value of t under $\mathbf{s}$). It is clear that the condition (ω) is met only by some languages of the considered kind. But we may prove that if the condition (ω) does hold in the case of some language, each simple yes-no question of this language is reducible to a set of atomic yes-no questions.

In order to continue we need the concept of prenex normal form of a d-wff: this concept is understood here in the standard sense. It is a well-known fact that for each d-wff there exists a logically equivalent d-wff in prenex normal form which contains the same free variables as the initial d-wff. Since logical entailment yields entailment in a language, then for each sentence A there exists a sentence B in prenex normal form such that $A \models B$ and $B \models A$. One can easily prove

LEMMA 2. *If A is a sentence and B is a sentence in prenex normal form such that $A \models B$ and $B \models A$, then the question* $?\,\{A, \neg A\}$ *is reducible to a set of questions* Φ *iff the question* $?\,\{B, \neg B\}$ *is reducible to the set* Φ.

Now we shall prove

LEMMA 3. *If the following condition holds*:

(ω) *for each sentential function Ax with exactly one free variable*, $\exists x\, Ax \,\|\!\!\models \mathbf{S}(Ax)$

then for each sentence B in prenex normal form, the simple yes-no question $?\,\{B, \neg B\}$ *is reducible to some set of atomic yes-no questions*.

PROOF: Let B be a sentence in prenex normal form. As above, let us observe that the clauses (i) and (iii) of the definition of reducibility are

[6] Let us recall here, however, that the set of closed terms of $\mathscr{L}$ need not be (but of course can be) infinite – we only imposed the non-emptiness condition on it.

fulfilled by any set of atomic yes-no questions with respect to the question ? $\{B, \neg B\}$. So it remains to be proved that for each sentence B in prenex normal form there exists a set of atomic yes-no questions Φ such that for each $\mu(\Phi)$-set Y, Y entails the sentence B or the sentence $\neg B$. The proof will go on by induction on the number of layers of quantifiers in the prefix of B.

Assume that the sentence B contains no layers of quantifiers. Since B is in prenex normal form, it follows that B is a quantifier-free sentence. So by Lemma 1 the question ? $\{B, \neg B\}$ is reducible to some set Φ of atomic yes-no questions and thus for each $\mu(\Phi)$-set Y, Y entails the sentence B or the sentence $\neg B$.

Assume now that B contains n layers of quantifiers in its prefix, where $n > 0$. By induction hypothesis for each sentence C in prenex normal form that contains n-1 layers of quantifiers in its prefix there exists a set of atomic yes-no questions Σ such that for each $\mu(\Sigma)$-set X, X entails C or entails $\neg C$.

We have four possibilities:

(a) B is of the form $\forall xD$, where D is in prenex normal form, contains n-1 layers of quantifiers in its prefix and x is not free in D,

(b) (b) B is of the form $\forall xD$, where D is in prenex normal form, contains n-1 layers of quantifiers in its prefix and x is free in D,

(c) (c) B is of the form $\exists xD$, where D is in prenex normal form, contains n-1 layers of quantifiers in its prefix and x is not free in D,

(d) (d) B is of the form $\exists xD$, where D is in prenex normal form, contains n-1 layers of quantifiers in its prefix and x is free in D.

If the possibility (a) holds, then – since B is a sentence – D is also a sentence; moreover, we have $D \models B$ as well as $B \models D$ and thus $\neg D \models \neg B$. The sentence D contains exactly n-1 layers of quantifiers and is in prenex normal form. Thus, by induction hypothesis there exists a set of of atomic yes-no questions, say, Φ, such that for each $\mu(\Phi)$-set Y, Y entails the sentence D or the sentence $\neg D$. Therefore for each $\mu(\Phi)$-set Y, Y entails the sentence B or the sentence $\neg B$

Suppose that the possibility (b) takes place. Now D is a sentential function with x as the only free variable; let us designate it by Dx. Let us now introduce the set $\mathbf{S}(Dx)$, i.e., the set of sentences of the form $D(x/t)$, where t is a closed term. Since the set of closed terms is nonempty, so is the set $\mathbf{S}(Dx)$; moreover, this set is made up of sentences in prenex normal form which contain exactly n-1 layers of quantifiers in their prefixes. So by induction hypothesis for each sentence C in the set $\mathbf{S}(Dx)$

there exists a set Ψ of atomic yes-no questions such that each $\mu(\Psi)$-set entails the sentence C or the sentence $\neg C$. Let us now consider a union of such sets of questions; to be more precise, for each $C \in \mathbf{S}(Dx)$ choose exactly one set of atomic yes-no questions which fulfill the inductive hypothesis and consider the set, say, Φ, which is the union of these sets. Let Y be a $\mu(\Phi)$-set. There are two possibilities: (1) the set Y entails each sentence from the set $\mathbf{S}(Dx)$, i.e., each sentence of the form $D(x/t)$, where t is a closed term; (2) the set Y entails some sentence of the form $\neg D(x/t)$. In the case of (2) Y entails the sentence $\neg B$. Let us now consider the case (1). Suppose that Y does not entail the sentence B, i.e., the sentence $\forall x\, Dx$. So there is a normal interpretation $\mathcal{J}$ such that $\mathcal{J}$ is a model of Y and the sentence $\exists x \neg Dx$ is true in $\mathcal{J}$. By assumption we have $\exists x \neg Dx \Vdash \mathbf{S}(\neg Dx)$. So at least one sentence of the form $\neg D(x/t)$ is true in $\mathcal{J}$. But Y entails each sentence of the form $D(x/t)$. Therefore each such sentence is true in $\mathcal{J}$. We arrive at a contradiction. So Y entails the sentence B. Thus we may say that each $\mu(\Phi)$-set entails the sentence B or the sentence $\neg B$.

If the possibility (c) holds, we proceed as in the case of (a).

Suppose finally that the possibility (d) takes place. Again, D is now a sentential function with x as the only free variable. We designate it by Dx. Then we construct the set Φ as in the case of (b). Let Y be a $\mu(\Phi)$-set. There are two possibilities: (1) the set Y entails some sentence from the set $\mathbf{S}(Dx)$, i.e., some sentence of the form $D(x/t)$, where t is a closed term; (2) the set Y entails each sentence of the form $\neg D(x/t)$. In the case of (1) Y entails the sentence B. Let us now consider the case (2). Suppose that Y does not entail the sentence $\neg\exists x\, Dx$. So there is a normal interpretation $\mathcal{J}$ such that $\mathcal{J}$ is a model of Y and the sentence $\exists x\, Dx$ is true in $\mathcal{J}$. By assumption we have $\exists x Dx \Vdash \mathbf{S}(Dx)$. So at least one sentence of the form $D(x/t)$ is true in $\mathcal{J}$. But Y entails each sentence of the form $\neg D(x/t)$. Therefore each such sentence is true in $\mathcal{J}$. We arrive at a contradiction. So Y entails the sentence $\neg\exists x\, Dx$. This sentence, however, is equal to $\neg B$. Thus we may say that each $\mu(\Phi)$-set entails the sentence B or the sentence $\neg B$.

■

Next we shall prove

THEOREM 5. *If the following condition holds*:

($\exists$) *for each sentential function Ax with exactly one free variable*, $\exists x\, Ax \Vdash \mathbf{S}(Ax)$

then each simple yes-no question is reducible to some set of atomic yes-no questions.

PROOF: Let ? $\{A, \neg A\}$ be a simple yes-no question. It is obvious that there exists a simple yes-no question ? $\{B, \neg B\}$ such that B is in prenex normal form and $A \models B$ as well as $B \models A$. By Lemma 3 the question ? $\{B, \neg B\}$ is reducible to some set of atomic yes-no questions; so by Lemma 2 the question ? $\{A, \neg A\}$ is reducible to some set of atomic yes-no questions.

■

Now we need the following

THEOREM 6. *Each safe question is reducible to some set of questions made up of simple yes-no questions*.

The proof of Theorem 6 is similar to that of Theorem 2. For details, see Wiśniewski (1994). By means of Theorem 5 and Theorem 6 we can prove

THEOREM 7. *If the following condition holds*:

(ω) *for each sentential function Ax with exactly one free variable*, $\exists x\, Ax \Vdash \mathbf{S}(Ax)$

then each safe question is reducible to some set of atomic yes-no questions.

PROOF: Let Q be a safe question. By Theorem 6 Q is reducible to some set of simple yes-no questions. Let Φ be an arbitrary but fixed set of simple yes-no questions such that Q is reducible to Φ. By Theorem 5 each question in Φ is reducible to some set of atomic yes-no questions. Let us then pair each question in Φ with exactly one (arbitrary but fixed) set of atomic yes-no questions to which the considered question in Φ is reducible. Let Ψ be the union of sets of atomic yes-no questions associated with the questions of Φ in the above manner. The set Ψ is a homogenous set of atomic yes-no questions. Then we proceed as in the proof of Theorem 3.

■

Thus, if the condition (ω) holds, then each safe question is reducible to some set of atomic yes-no questions. But it is not the case that the condition (ω) holds for any language of the considered kind. So designing the semantics in such a way that the condition (ω) would be met is the price which, if paid, gives us the unrestricted reducibility of safe questions to homogenous sets of atomic yes-no questions. And it may be a high price: as a by-product we may obtain the lack of compactness of entailment as well as of mc-entailment.[7]

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