# Strong Entailments and Minimally Inconsistent Sets. Research Report

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# *Strong Entailments and Minimally Inconsistent Sets*

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**Abstract** Some non-monotonic relations of single- and multiple-conclusion entailment based on minimally inconsistent sets are defined and analyzed. A proof-theoretic account of minimally inconsistent sets is provided. Applications concerning minimal disjunctions, contraction, and argument analysis are presented.

## 1 Introduction

**1.1 Single-conclusion Entailment** The idea of *transmission of truth* underlies the intuitive concept of entailment. According to the idea, entailment is akin to an input-output device which, when fed with truth at the input, gives truth at the output. The input need not consist of truths, but *if* it does, it transforms into a true output. Similarly, *if* the premises are all true, any conclusion entailed by them must be true, although the truth of premises is not a necessary condition for entailment to hold. Or, to put it differently, the *hypothetical* truth of premises warrants the truth of an entailed conclusion.[1]

Logicians operate with well-formed formulas (wffs for short) and conceptualize entailment as a semantic relation between sets of wffs and single wffs. At the same time they tend to understand the 'if' above in the sense of material conditional. Yet, since a material conditional with false antecedent is true irrespective of the logical value of the consequent, as a consequence one gets:

(I) *a set of wffs which cannot be simultaneously true, i.e. an inconsistent set, entails every wff.*

Moreover, a material conditional with true consequent is true irrespective of the logical value of the antecedent, and hence:

(II) *a logically valid wff is entailed by any set of wffs.*

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Both (I) and (II) are a kind of by-products and we have adjusted to live with them. But (I) as well as (II) seem to contravene the intuitive idea of transmission of truth. To say that "truth is transmitted" seems to presuppose that it *can* occur at the input and that it *need not* occur at the output.

Another drawback of the received view is this. To say that the hypothetical truth of sentences in a set $\Phi$ warrants the truth of a sentence $\psi$ seems to presuppose that the truth of *all* the sentences in $\Phi$ is, jointly, *relevant* to the truth of $\psi$. Entailment intuitively construed is a kind of semantic entrenchment of an entailed sentence in a set of sentences that entails it: a set of sentences $\Phi$ that entails a sentence $\psi$ comprises neither less nor more sentences than those the hypothetical truth of which, jointly, warrants the truth of $\psi$. On the other hand, entailment defined in the usual way, by using, inter alia, the material 'if', is monotone:

**(M):** *a wff $B$ entailed by a set of wffs $X$ is entailed by any superset of $X$ as well*

and hence the wff $B$ is also entailed by sets of wffs which contain elements that are irrelevant with regard to the transmission of truth and/or the semantic entrenchment effect(s): their hypothetical truth do not contribute in any way to the truth of $B$.

**1.2 Multiple-conclusion Entailment** Entailment is a relation between a set of sentences/wffs on the one hand, and a single sentence/wff on the other; for clarity, let us call it *sc-entailment* (after "single-conclusion entailment"). Multiple-conclusion entailment (*mc-entailment* for short) is a relation between sets of wffs. Non-singleton conclusion sets are allowed. The underlying idea is: an mc-entailed set must contain at least one true wff *if* the respective mc-entailing set consists of truths. Or, to put it differently, the hypothetical truth of all the wffs in an mc-entailing set warrants the existence of a true wff in the mc-entailed set.[2]

Mc-entailment can hold for trivial reasons: $X$ mc-entails $Y$ because $X$ sc-entails at least one wff in $Y$. But mc-entailment can also hold non-trivially: it happens that a set of wffs, $X$, mc-entails a set of wffs, $Y$, although $X$ does not sc-entail any wff in $Y$. For instance (taking Classical Propositional Logic as the basis), the truth of all the wffs in the set $X = \{p \rightarrow q \vee r, p\}$ warrants the existence of a true wff in the set $Y = \{q, r\}$, but neither $q$ nor $r$ is sc-entailed by $X$ or, to put it differently, the hypothetical truth of the wffs in $X$ warrants that at least one of: $q$, $r$ is true, but neither warrants the truth of $q$ nor the truth of $r$.

The concept of mc-entailment is more general than that of sc-entailment. One can always define sc-entailment as mc-entailment of a singleton set. However, it is not the case that mc-entailment can always be defined in terms of sc-entailment.[3]

Mc-entailment explicated by means of the material 'if' suffers similar drawbacks to those of sc-entailment explicated in this way:

(I') *any set of wffs is mc-entailed by an inconsistent set of wffs,* and
(II') *a set of wffs that contains a logically valid wff is mc-entailed by any set of wffs*.

Mc-entailment is *left-monotone*, that is:

**(LM):** *a set $Y$ which is mc-entailed by a set $X$ is also mc-entailed by any superset of $X$.*

Hence there exist mc-entailing sets of wffs which contain, inter alia, wffs that are semantically irrelevant w.r.t. the corresponding mc-entailed sets.

Mc-entailment is also *right-monotone*:

**(RM):** *if $X$ mc-entails $Y$, then $X$ mc-entails any superset of $Y$ as well.*

Thus there exist mc-entailed sets of wffs which have some elements that are semantically irrelevant w.r.t. the respective sets of wffs which mc-entail them. For instance, $\{p \to q \vee r, p\}$ mc-entails $\{q, r\}$ and hence the set $\{q, r, s\}$ as well.

**1.3 Aims of the Paper** One of the ways of thinking of mc-entailed sets is to construe them as items effectively *delimiting* search spaces: a set of wffs $Y$ mc-entailed by a set of wffs $X$ is a minimal set that comprises wffs among which a truth must lie if the wffs in $X$ are all true. 'Minimal' means here 'no proper subset of $Y$ behaves analogously w.r.t. $X$.' Another way of thinking about an mc-entailed set is to construe it as characterizing the *relevant cases* to be considered, for if $X$ mc-entails $Y$ and each wff in $Y$ sc-entails a wff $B$, the wff $B$ is sc-entailed by $X$ as well. However, mc-entailment is right-monotone and thus the standard concept of mc-entailment is too broad to reflect the above ideas.

In this paper we introduce and examine a concept that enables us to express these ideas. We dub it 'strong multiple-conclusion entailment.' Formally, strong mc-entailment is a subrelation of mc-entailment. We define strong mc-entailment in a way which allows us to avoid the drawbacks (I') and (II') indicated above. Moreover, strong mc-entailment is neither left-monotone nor right-monotone. As a by-product we get a concept of single-conclusion entailment which, in turn, is free of the drawbacks pointed out at the beginning of this paper. We coin the concept 'strong single-conclusion entailment.' This concept of sc-entailment is analyzed in the paper as well.

The paper is organized as follows.

In Section 2 we introduce the logical apparatus needed.

Section 3 is devoted to strong mc-entailment. Its definition is provided in subsection 3.1, and the adequacy issue in addressed therein. The consecutive subsections include propositions and theorems characterizing basic properties of strong mc-entailment. In particular, we examine the relation between strong mc-entailment and minimally inconsistent sets, and address the finiteness issue. It occurs that, as long as the underlying logic has the compactness property, strong mc-entailment holds *only* between finite sets of wffs. This allows us to explicate the intuitive notion of a 'minimal entailed disjunction' in terms of strong mc-entailment.

An analysis of strong sc-entailment is presented in Section 4. We define strong sc-entailment as the strong mc-entailment of a singleton set. It is proven that classical sc-entailment from a consistent set of premises boils down to strong sc-entailment from some finite subset(s) of the set: any wff classically sc-entailed by a consistent set is also strongly sc-entailed by a finite subset of the set. This sheds new light on the contraction issue; details are addressed in subsection 4.5 and in Appendix 1.

In Section 5 we present a proof-theoretic account of minimally inconsistent sets and thus, indirectly, of strong mc- and sc-entailments. It is shown that a subsystem of a system initially designed for other purposes is sufficient here; a proof of the key lemma that leads to the completeness result is presented in Appendix 2.

## 2 The Logical Basis

For simplicity, we remain at the propositional level only, and we consider the case of Classical Propositional Logic (hereafter: CPL). For clarity, we assume that CPL is worded is a language characterized as follows.

The vocabulary of the language comprises a countably infinite set `Var` of propositional variables, the propositional constant $\bot$ (*falsum*), the connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$, and brackets. The set `Form` of *wffs* of the language is the smallest set that includes $\mathtt{Var} \cup \{\bot\}$ and satisfies the following conditions: (1) if $A \in \mathtt{Form}$, then '$\neg A$' $\in \mathtt{Form}$; (2) if $A, B \in \mathtt{Form}$, then '$(A \otimes B)$' $\in \mathtt{Form}$, where $\otimes$ is any of the connectives: $\vee, \wedge, \rightarrow, \leftrightarrow$. We adopt the usual conventions concerning omitting brackets. We use $A, B, C, D$, with subscripts when needed, as metalanguage variables for wffs, and $X, Y, W, Z$, with or without subscripts, as metalanguage variables for sets of wffs. The letters $p, q, r, s, t$ are exemplary elements of `Var`.

By a proper superset of a set of wffs $X$ we mean any set of wffs $Z$ such that $X$ is a proper subset of $Z$.

Let $\mathbf{1}$ stand for truth and $\mathbf{0}$ for falsity. A CPL-*valuation* is a function $v : \mathtt{Form} \mapsto \{\mathbf{1}, \mathbf{0}\}$ satisfying the following conditions: (a) $v(\bot) = \mathbf{0}$; (b) $v(\neg A) = \mathbf{1}$ iff $v(A) = \mathbf{0}$; (c) $v(A \vee B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ or $v(B) = \mathbf{1}$; (d) $v(A \wedge B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ and $v(B) = \mathbf{1}$; (d) $v(A \rightarrow B) = \mathbf{1}$ iff $v(A) = \mathbf{0}$ or $v(B) = \mathbf{1}$; (e) $v(A \leftrightarrow B) = \mathbf{1}$ iff $v(A) = v(B)$. Remark that the domain of $v$ includes `Var`. Needless to say, there are (uncountably) many CPL-valuations.

For brevity, in what follows we omit the references to CPL. Unless otherwise stated, the semantic relations analyzed are supposed to hold between sets of CPL-wffs, or sets of CPL-wffs and single CPL-wffs, and by valuations we mean CPL-valuations.

We define:

**Definition 1 (Sc-entailment)** $X \models A$ *iff for each valuation* $v$*:*

- *if* $v(B) = \mathbf{1}$ *for every* $B \in X$*, then* $v(A) = \mathbf{1}$.

**Definition 2 (Logical equivalence)** $A \Leftrightarrow B$ *iff* $\{A\} \models B$ *and* $\{B\} \models A$.

**Definition 3 (Consistency, inconsistency, and validity)** *A set of wffs* $X$ *is* consistent *iff there exists a valuation* $v$ *such that for each* $A \in X$, $v(A) = \mathbf{1}$*; otherwise* $X$ *is* inconsistent. *A wff* $B$ *is* inconsistent *iff the singleton set* $\{B\}$ *is inconsistent. A wff* $B$ *is* valid *iff for each valuation* $v$, $v(B) = \mathbf{1}$.

**Definition 4 (Mc-entailment)** $X \,||\!\!= Y$ *iff for each valuation* $v$*:*

- *if* $v(B) = \mathbf{1}$ *for every* $B \in X$*, then* $v(A) = \mathbf{1}$ *for at least one* $A \in Y$.

We will also need the well-known concept of minimally inconsistent set:[4]

**Definition 5 (Minimally inconsistent set; MI-set)** *A set of wffs* $X$ *is* minimally inconsistent *iff* $X$ *is inconsistent but each proper subset of* $X$ *is consistent.*

***Terminology***. For brevity, we will be referring to minimally inconsistent sets as to MI-*sets*.

Note that $\emptyset$ is not a MI-set. Singleton MI-sets have inconsistent wffs as the (only) elements. Here are examples of non-singleton MI-sets:

$$\{p, \neg p\} \tag{1}$$

$$\{p \vee q, \neg p, \neg q\} \tag{2}$$

$$\{p \rightarrow q, p, \neg q\} \tag{3}$$

$$\{p \rightarrow q \vee r, p, \neg q, \neg r\} \tag{4}$$

$$\{p \rightarrow q, q \rightarrow r, \neg(p \rightarrow r)\} \tag{5}$$

Clearly, the following holds:

**Corollary 1** *$X$ is a MI-set iff $X$ is inconsistent and for each $A \in X$, the set $X \setminus \{A\}$ is consistent.*

We are dealing here with CPL and hence the following *compactness claim* holds:

(♣) *for each set of wffs Z: the set Z is consistent iff each finite subset of Z is consistent.*

Hence the following is true:

**Lemma 1** *Each MI-set is finite.*

**Proof** Suppose that $Y$ is an infinite MI-set. Then $Y$ is inconsistent and each proper subset of $Y$ is consistent. Hence each finite subset of $Y$ is consistent and hence, by (♣), $Y$ is consistent. A contradiction. □

As the proof of Lemma 1 shows, the finiteness of MI-sets is a direct consequence of the compactness claim (♣). Thus as long as a logic is compact, the respective MI-sets are finite. However, there are logics for which the analogues of (♣) do not hold and thus finiteness is not a property of MI-sets in general.[5]

## 3 Strong Mc-entailment

**3.1 Definition and the Adequacy Issue** We use $\Vdash\!\prec$ as the symbol for strong mc-entailment, and we define the relation as follows:

**Definition 6 (Strong mc-entailment)** $X \Vdash\!\prec Y$ *iff*

1. $X \Vert\!\models Y$, *and*
2. *for each* $A \in X : (X \setminus \{A\}) \Vert\!\not\models Y$, *and*
3. *for each* $B \in Y : X \Vert\!\not\models (Y \setminus \{B\})$.

The consecutive clauses of the above definition express the following intuitions: the hypothetical truth of all the wffs in $X$ warrants the existence of at least one true wff in $Y$, yet the warranty disappears as $X$ decreases or $Y$ decreases. In other words, $X$ and $Y$ are minimal sets under the warranty provided by the clause 1.

Here are simple examples:

$$\{p\} \Vdash\!\prec \{p\} \tag{6}$$
$$\emptyset \Vdash\!\prec \{p, \neg p\} \tag{7}$$
$$\{p, \neg p\} \Vdash\!\prec \emptyset \tag{8}$$
$$\emptyset \Vdash\!\prec \{p \vee \neg p\} \tag{9}$$
$$\{\neg p, \neg q, p \vee q\} \Vdash\!\prec \emptyset \tag{10}$$
$$\emptyset \Vdash\!\prec \{p, q, \neg(p \vee q)\} \tag{11}$$
$$\{p \vee q\} \Vdash\!\prec \{p, q\} \tag{12}$$
$$\{p \to q \vee r, p\} \Vdash\!\prec \{q, r\} \tag{13}$$
$$\{p \wedge q \to r, \neg r\} \Vdash\!\prec \{\neg p, \neg q\} \tag{14}$$
$$\{p \vee (q \vee r)\} \Vdash\!\prec \{q \vee r, p \vee r, p \vee q\} \tag{15}$$
$$\{\neg(p \wedge (q \wedge r))\} \Vdash\!\prec \{\neg q \vee \neg r, \neg p \vee \neg r, \neg p \vee \neg q\} \tag{16}$$

Note that $\emptyset \Vdash\!\not\prec \emptyset$, since $\emptyset \Vert\!\not\models \emptyset$.

Clearly, $\Vdash\!\prec$ is neither left-monotone nor right-monotone.

Observe that the following are true:[6]

$$\{p, \neg p\} \;||\!\not\prec\; \{q\} \tag{17}$$

$$\{p\} \;||\!\not\prec\; \{p \vee \neg p\} \tag{18}$$

Thus it is neither the case that any inconsistent set of wffs strongly mc-entails any set of wffs nor it is the case that a set which involves a valid wff is strongly mc-entailed by any set of wffs. Hence strong mc-entailment is free of the drawbacks (I') and (II') pointed out in Section 1.2.

**3.2 Basic Properties of Strong Mc-entailment** Let us now prove:

**Theorem 1** *$X \;||\!\prec Y$ iff $X \;||\!= Y$ and the following conditions hold:*

1. *there is no proper subset $Z$ of $X$ such that $Z \;||\!= Y$,*
2. *there is no proper subset $W$ of $Y$ such that $X \;||\!= W$.*

**Proof**
($\Rightarrow$) Assume that $X \;||\!\prec Y$. Hence $X \;||\!= Y$.

If $X = \emptyset$, then $X$ has no proper subset. Similarly, if $Y = \emptyset$, then $Y$ has no proper subset.

Assume that $X$ is a singleton set. Let $X = \{C\}$. The only proper subset of $X$ is thus $\emptyset$. But, due to the clause 2 of Definition 6, $(X \setminus \{C\}) \;||\!\neq Y$. On the other hand, $(X \setminus \{C\}) = \emptyset$ and hence $\emptyset \;||\!\neq Y$.

Now assume that $X$ is neither the empty set nor a singleton set. Let $Z$ be a proper subset of $X$.

Assume that $Z = \emptyset$. Suppose that $Z \;||\!= Y$. Hence $\emptyset \;||\!= Y$ and thus $(X \setminus \{A\}) \;||\!= Y$ for any $A \in X$. Therefore $X \;||\!\not\prec\; Y$. We arrive at a contradiction. Thus $Z \;||\!\neq Y$.

Now assume that $Z \neq \emptyset$. Clearly, $Z$ is a subset of some set of the form $X \setminus \{A\}$, where $A \in X$. Therefore, by the clause 2 of Definition 6, $Z \;||\!\neq Y$.

We reason similarly in the case of $Y$.

($\Leftarrow$) The only cases worth some attention are: (a) $X = \emptyset$ and (b) $Y = \emptyset$. But if (a) holds, the clause 2 of Definition 6 is trivially fulfilled, and similarly for (b) and the clause 3. □

According to Theorem 1, $X$ strongly mc-entails $Y$ just in case $X$ mc-entails $Y$, but does not mc-entail any proper subset of $Y$, and $Y$ itself is not mc-entailed by any proper subset of $X$.

As immediate consequences of Theorem 1 one gets:

**Proposition 1** *If $X \;||\!\prec Y$, then:*

1. *$Z \;||\!\not\prec\; Y$, where $Z$ is either a proper subset or a proper superset of $X$,*
2. *$X \;||\!\not\prec\; W$, where $W$ is either a proper subset of a proper superset of $Y$.*

**Proposition 2** *If $X \;||\!\prec Y$, then each proper subset of $X$ is consistent.*

Note that Proposition 2 does not require each strongly mc-entailing set be consistent. As a matter of fact, some strongly mc-entailing sets are inconsistent. These sets, however, strongly mc-entail the empty set only. This is due to:

**Proposition 3** *If $X \;||\!\prec Y$ and $X$ is inconsistent, then $Y = \emptyset$.*

**Proof** Suppose that $X \Vdash\!\prec Y$ and $X$ is an inconsistent set, but $Y \neq \emptyset$. There are two possibilities: (i) $Y = \{D\}$ for some wff $D$, or (ii) $Y = W \cup \{D\}$ for some wff $D$ and a non-empty set of wffs $W$.

If (i) is the case, then $X \Vdash\!\prec \{D\}$. It follows that $X \not\Vdash\!\models \emptyset$. However, as $X$ is inconsistent, we also have $X \Vdash\!\models \emptyset$. A contradiction.

If (ii) holds, we have $X \Vdash\!\prec (W \cup \{D\})$ and therefore $X \not\Vdash\!\models W$. But $X$ is inconsistent and thus $X \Vdash\!\models W$. A contradiction again. □

According to Proposition 3, an inconsistent set strongly mc-entails, if any[7], only the empty set, or, to put it differently, there is no non-empty set which is strongly mc-entailed by an inconsistent set. It follows that an inconsistent set does not strongly mc-entail even the singleton set $\{\bot\}$.

Observe that singleton sets which have inconsistent wffs as elements are not strongly mc-entailed at all. This is due to:

**Proposition 4** *Let $C$ be an inconsistent wff. There is no set of wffs $X$ such that $X \Vdash\!\prec \{C\}$.*

**Proof** Suppose that $X \Vdash\!\prec \{C\}$ for some set of wffs $X$. Then $X$ will be both inconsistent (since $X \Vdash\!\models \{C\}$ holds) and consistent (because $X \not\Vdash\!\models \emptyset$ would hold). □

However, the following is true as well:

**Proposition 5** *There exist inconsistent sets of wffs which are strongly mc-entailed by some sets of wffs.*

Examples (7) and (11) presented above support the claim of Proposition 5. Here is an example that does not involve the empty set:[8]

$$\{\neg q\} \Vdash\!\prec \{\neg p, p \wedge \neg q\} \tag{19}$$

Let $W$ be a finite set of wffs. By $\bigvee W$ we mean a disjunction of all the wffs in $W$; if $W = \{A\}$, then $\bigvee W = A$, and if $W = \emptyset$, then we put $\bigvee W = \bot$. One can easily prove:

**Proposition 6** *If $X \Vdash\!\prec Y$, then:*

1. *no wff in $X$ is valid,*
2. *for each non-empty and finite proper subset $W$ of $Y$: $\bigvee W$ is not valid.*

**Proof** (1) holds due to the clause 2 of Definition 6. As for (2), suppose otherwise. Then $X \Vdash\!\models W$ for some finite and non-empty proper subset of $W$ of $Y$ (namely, for the set of disjuncts of $\bigvee W$) and therefore, by Theorem 1, $X \not\Vdash\!\prec Y$. A contradiction. □

Hence a set of wffs which contains a valid wff does not strongly mc-entail any set of wffs, and it never happens that that a disjunction of all the elements of a non-empty *proper subset* of a strongly mc-entailed set is valid. However, it happens that a disjunction of *all* the wffs of a strongly mc-entailed set is valid; examples (7) and (11) are cases in point here. What is more, Proposition 6 still leaves room for the effect described by:

**Proposition 7** *Let $C$ be a valid wff. Then $X \Vdash\!\prec \{C\}$ iff $X = \emptyset$.*

**Proof** Let $C$ be a valid wff.
($\Rightarrow$) Suppose that $X \Vdash\!\prec \{C\}$, but $X \neq \emptyset$. There are two possibilities: (i) $X = \{A\}$ for some wff $A$ or (ii) $X = Y \cup \{A\}$ for some wff $A$ and a non-empty set of wffs $Y$.

Assume that (i) is the case. Thus $\{A\} \Vert\!\!\prec \{C\}$. Therefore, by the clause 2 of Definition 6, $\emptyset \Vert\!\!\not\models \{C\}$. But, by the initial assumption, $C$ is valid and hence $\emptyset \Vert\!\!\models \{C\}$. A contradiction.

Now assume that (ii) holds. We have $(Y \cup \{A\}) \Vert\!\!\prec \{C\}$ and therefore $Y \Vert\!\!\not\models \{C\}$. But $C$ is valid and thus $Y \Vert\!\!\models \{C\}$. A contradiction again.

($\Leftarrow$) If $X = \emptyset$, the clause 2 of Definition 6 is trivially fulfilled. At the same time we have $\emptyset \Vert\!\!\models \{C\}$ and $\emptyset \Vert\!\!\not\models \emptyset$. □

Thus valid wffs can occur as elements of strongly mc-entailed sets, but these sets are singleton sets which, moreover, are strongly mc-entailed only by the empty set.

Observe that strongly mc-entailing sets do not contain logically equivalent wffs, and similarly for strongly mc-entailed sets. To be more precise, the following holds:

**Proposition 8** *If $X \Vert\!\!\prec Y$, then:*

1. *there are no $A, B \in X$ such that $A \Leftrightarrow B$,*
2. *there are no $C, D \in Y$ such that $C \Leftrightarrow D$.*

Let us also note:

**Proposition 9** *If $X$ has at least two elements, then $X \Vert\!\!\not\prec X$.*

**Proof** Suppose otherwise. Let $A \in X$. Then $(X \setminus \{A\}) \Vert\!\!\not\models X$. But $((X \setminus \{A\}) \cap X) \neq \emptyset$ and hence $(X \setminus \{A\}) \Vert\!\!\models X$. A contradiction. □

**Proposition 10** *$\{A\} \Vert\!\!\prec \{A\}$ iff $A$ is neither valid nor inconsistent.*

**Proof** Clearly, $\{A\} \Vert\!\!\models \{A\}$, and $\{A\} \setminus \{A\} = \emptyset$. Observe that $A$ is not valid iff $\emptyset \Vert\!\!\not\models \{A\}$, and $A$ is not inconsistent iff $\{A\} \Vert\!\!\not\models \emptyset$. □

**3.3 Reductions to Minimally Inconsistent Sets** Let $\neg_n$ stand for an $n$-element ($n \geq 0$) string of sentential negations, $\neg$. We need the following auxiliary notion.

**Definition 7 (Complementary wff)**

1. *If $A$ is of the form $\neg_n C$, where $n$ is even, then $\overline{A}$ is $\neg_{n+1} C$.*
2. *If $A$ is of the form $\neg_n C$, where $n$ is odd, then $\overline{A}$ is $\neg_{n-1} C$.*
3. *$\overline{Z} = \{\overline{C} : C \in Z\}$.*

**Corollary 2** *$\overline{\emptyset} = \emptyset$, and $\overline{(\overline{A})} = A$. Moreover, $\overline{A} \in Y$ iff $A \in \overline{Y}$.*

**Corollary 3** *$X \Vert\!\!\models Y$ iff $\emptyset \Vert\!\!\models Y \cup \overline{X}$ iff $X \cup \overline{Y} \Vert\!\!\models \emptyset$.*

**Corollary 4** *For each wff $A$ there exists a wff $B$ such that $A = \overline{B}$.*

Once a wff $A$ is given, one can easily "calculate" the wff $B$ such that $A = \overline{B}$. Each wff $A$ falls under the schema $\neg_n C$, where $n$ is either even or odd. The corresponding $B$ can be calculated by means of the following principles:

**(even):** If $A = \neg_n C$ and $n$ is even, then $B = \neg_{n+1} C$.
**(odd):** If $A = \neg_n C$ and $n$ is odd, then $B = \neg_{n-1} C$.

For example, $p = \overline{\neg p}$, $\neg p = \overline{p}$, $p \rightarrow p = \overline{\neg(p \rightarrow p)}$, and $\neg(p \rightarrow p) = \overline{(p \rightarrow p)}$.

Strong mc-entailment and MI-sets are linked in the following way:

**Theorem 2** *$X \Vert\!\!\prec Y$ iff $X \cap \overline{Y} = \emptyset$ and $X \cup \overline{Y}$ is a* MI*-set.*

**Proof**
($\Rightarrow$) Let $X \Vert\!\prec Y$. Suppose that $X \cap \overline{Y} \neq \emptyset$. Let $A \in X$ and $A \in \overline{Y}$. Thus $\overline{A} \in Y$ and $Y = Y^* \cup \{\overline{A}\}$, where $\overline{A} \notin Y^*$. As $X \Vert\!\prec Y$ holds, we have $X \Vert\!\models (Y^* \cup \{\overline{A}\})$ and hence $(X \cup \{A\}) \Vert\!\models Y^*$. But, since $A \in X$, $(X \cup \{A\}) = X$. Therefore $X$ mc-entails the proper subset $Y^*$ of $Y$ and thus $X \Vert\!\not\prec Y$. A contradiction. Hence $X \cap \overline{Y} = \emptyset$.

If $X \Vert\!\prec Y$, then $X \Vert\!\models Y$ and thus the set $X \cup \overline{Y}$ is inconsistent. Let us designate $X \cup \overline{Y}$ by $Z$.

If $A \in Z$, then $A \in X$ or $A \in \overline{Y}$.

Assume that $A \in X$. By the clause 2 of Definition 6, $(X \setminus \{A\}) \Vert\!\not\models Y$ and thus the set $(X \setminus \{A\}) \cup \overline{Y}$ is consistent, that is, $(Z \setminus \{A\})$ is consistent.

Now assume that $A \in \overline{Y}$. It follows that $\overline{A} \in Y$. By the clause 3 of Definition 6, $X \Vert\!\not\models (Y \setminus \{\overline{A}\})$ and hence the set $X \cup (\overline{Y} \setminus \{A\})$ is consistent, that is, the set $(Z \setminus \{A\})$ is consistent.

Thus, by Corollary 1, $X \cup \overline{Y}$ is a MI-set.
($\Leftarrow$) Assume that $X \cap \overline{Y} = \emptyset$ and that $X \cup \overline{Y}$ is a MI-set. From the latter it follows that $X \Vert\!\models Y$.

Again, let $Z = X \cup \overline{Y}$.

Suppose that $(X \setminus \{A\}) \Vert\!\models Y$ for some $A \in X$. Then the set:

$$(X \setminus \{A\}) \cup \overline{Y} \tag{20}$$

is inconsistent. Yet, since $X \cap \overline{Y} = \emptyset$, the set (20) is a proper subset of $Z$. Thus $Z$ is not a MI-set. A contradiction.

Now suppose that $X \Vert\!\models (Y \setminus \{B\})$ for some $B \in Y$. Let us designate $(Y \setminus \{B\})$ by $Y^*$. As $X \Vert\!\models Y^*$ holds, the set:

$$X \cup \overline{Y^*} \tag{21}$$

is inconsistent. But $X \cap \overline{Y} = \emptyset$ and hence the set (21) is a proper subset of $Z$. Thus $Z$ is not a MI-set. A contradiction again.

Therefore $X \Vert\!\prec Y$. □

**Remark 1** As the second part of the proof of Theorem 2 shows, one can get $X \Vert\!\prec Y$ from the fact that $X \cup \overline{Y}$ is a MI-set *on the condition* that $X \cap \overline{Y} = \emptyset$ holds. The first part of the proof shows, in turn, that this is a necessary condition for $X \Vert\!\prec Y$ to hold. For example, let $X = \{p \vee q, \neg p, \neg q\}$ and $Y = \{p, q\}$. Then $\overline{Y} = \{\neg p, \neg q\}$ and hence $X \cup \overline{Y}$ is a MI-set. However, $X \Vert\!\not\prec Y$, since $\{p \vee q\} \Vert\!\models \{p, q\}$. On the other hand, $(X \cap \overline{Y}) = \{\neg p, \neg q\} \neq \emptyset$.

The following is true as well:

**Proposition 11** $X \Vert\!\prec \overline{Y}$ *iff* $X \cap Y = \emptyset$ *and* $X \cup Y$ *is a* MI-*set.*

**Proof** By Theorem 2. It suffices to observe that $\overline{(\overline{Y})} = Y$. □

Let us also note:

**Proposition 12** $X \Vert\!\prec \emptyset$ *iff* $X$ *is a* MI-*set.*

**Proof** By Theorem 2. □

**Proposition 13** $\emptyset \Vert\!\prec Y$ *iff* $\overline{Y}$ *is a* MI-*set.*

**Proof** By Theorem 2. □

**Proposition 14** $\emptyset \Vert\!\prec \overline{Y}$ *iff* $Y$ *is a* MI-*set.*

**Proof** By Proposition 11. □

**3.4 Strict Finiteness and Minimal Disjunctions** Let us now prove:

**Theorem 3** *If $X \parallel\!\!\prec Y$, then $X$ and $Y$ are finite sets.*

**Proof** If $X \parallel\!\!\prec Y$, then, by Theorem 2, $X \cup \overline{Y}$ is a MI-set and therefore, by Lemma 1, a finite set. Thus both $X$ and $Y$ are finite sets as well. □

Theorem 3 states that as long as CPL is concerned, strong mc-entailment is *strictly finitistic*: it holds only between finite sets of wffs. As the proof shows, this is due mainly to Lemma 1 which, in turn, relies upon the compactness claim (♣). Thus, generally speaking, strong mc-entailment *in a logic* is strictly finitistic if the respective compactness claim holds for the logic. However, there are logics for which the analogues of (♣) do not hold. Strong mc-entailment in these logics need not be strictly finitistic.

Yet, in the case of CPL strong mc-entailment *is* strictly finitistic and this is a virtue rather than a vice. Leaving aside the obvious computational benefits, let us only mention one conceptual gain.

We often use the notion of 'minimal disjunction entailed by a set of wffs' or its syntactic counterparts.[9] The concept is deeply relational and can be explicated as follows: a wff $D$ is a minimal disjunction entailed by a set of wffs $X$ if: (a) $D$ has the form of a disjunction of two or more wffs, and (b) $D$ is sc-entailed by $X$, but not by any proper subset of $X$, and (c) no disjunction of some, but not all disjuncts of $D$ is sc-entailed by $X$. However, for any *finite* sets $X$, $Y$ of (CPL)wffs we have:

$$X \parallel\!\!\models Y \quad \textit{iff} \quad X \models \bigvee Y \tag{22}$$

Thus theorems 3 and 1 imply:

**Corollary 5** *Let $Y$ be an at least two-element set of wffs. A set of wffs $X$ strongly mc-entails the set $Y$ iff $\bigvee Y$ is a minimal disjunction entailed by $X$.*

Therefore, as long as CPL (or any other logic having the compactness property) is concerned, one can identify minimal disjunctions entailed by a finite set of wffs $X$ with disjunctions of all the elements of (at least two-member) sets strongly mc-entailed by $X$. Moreover, once we have a proof-theoretic account of strong mc-entailment, we also have a proof-theoretic account of the phenomenon of arriving at minimal disjunctions.

## 4 Strong Sc-entailment

Sc-entailment traditionally construed can be identified with mc-entailment of a singleton set. Similarly, we define strong sc-entailment as strong mc-entailment of a singleton set. We use $\mid\!\prec$ as the symbol for strong sc-entailment.

**Definition 8 (Strong sc-entailment)** $X \mid\!\prec C$ *iff* $X \parallel\!\!\prec \{C\}$.

As an immediate consequence of Definition 8 and Theorem 2 one gets:

**Theorem 4** $X \mid\!\prec A$ *iff* $\overline{A} \notin X$ *and* $X \cup \{\overline{A}\}$ *is a* MI-*set.*

**Remark 2** Note that the transition from right to left requires $\overline{A} \notin X$ to hold. For example, the following:

$$\{p, p \to \neg q, q\} \cup \{\overline{\neg q}\} \tag{23}$$

is a MI-set.[10] On the other hand, the following:

$$\{p, p \to \neg q, q\} \mathrel{|\!\!\!<} \neg q$$

*does not hold*, since $\{p, p \to \neg q\} \models \neg q$. But, as $\overline{\neg q} = q$, the condition $\overline{A} \notin X$ is violated.

In general, if $X \cup \{\overline{A}\}$ is a MI-set and $\overline{A} \in X$, then $X$ is an inconsistent set and thus, by Theorem 1, $X \not\Vdash \{A\}$ (since $X \Vdash \emptyset$) and hence $X \not\mathrel{|\!\!\!<} A$.

**4.1 Strong Sc-entailment and Lehrer's Notion of Relevant Deductive Argument** Strong sc-entailment is a special case of strong mc-entailment defined above. There are, however, some affinities between the concept of strong sc-entailment and the notion of relevant deductive argument introduced long ago by Keith Lehrer (cf. [6]). Here is Lehrer's definition:

> An argument *RD* is a relevant deductive argument if and only if *RD* contains a nonempty set of premises $P_1$, $P_2$, ..., $P_n$ and a conclusion $C$ such that a set of statements consisting of just $P_1, P_2, \ldots, P_n$, and $\neg C$ (or any truth functional equivalent of $\neg C$) is a minimally inconsistent set. A set of statements is a minimally inconsistent set if and only if the set of statements is logically inconsistent and such that every proper subset of the set is logically consistent. (Lehrer [6], p. 298.)

Is strong sc-entailment just the semantic relation that holds between premises and conclusions of Lehrer's relevant deductive arguments? The answer is: not exactly, but almost. As Theorem 4 and Remark 2 illustrate, the fact that $\{P_1, \ldots, P_n, \neg C\}$ is a MI-set is a necessary but insufficient condition for $\{P_1, P_2, \ldots, P_n\} \mathrel{|\!\!\!<} C$ to hold; it is also required that $\overline{C} \notin \{P_1, P_2, \ldots, P_n\}$. On the other hand, the expression "a set of statements consisting of just $P_1$, $P_2$, ..., $P_n$, and $\neg C$ (or any truth functional equivalent of $\neg C$)" seems to secure that the additional requirement is to be met. However, the definition of strong sc-entailment introduces a general semantic concept which may be used outside the realm of argument analysis, and does not impose any initial conditions on the respective $X$ and $A$. In particular, it is not presupposed that $X$ is non-empty, which is done (for obvious reasons) by Lehrer's definition. As we will see, there are wffs which are strongly sc-entailed by the empty set.

**4.2 Basic Properties of Strong Sc-entailment** Theorem 1 and Definition 8 yield:

**Theorem 5** $X \mathrel{|\!\!\!<} A$ *iff*

1. *$X$ is consistent, and*
2. *$X \models A$, and*
3. *for each proper subset $Z$ of $X$: $Z \not\models A$.*

Therefore strong sc-entailment is not monotone; as a matter of fact, it is "antimonotone," that is, the following holds:

**Proposition 15** *If $X \mathrel{|\!\!\!<} A$ and $X \subset Y$, where $Y \neq X$, then $Y \not\mathrel{|\!\!\!<} A$.*

**Proof** By Definition 8 and Proposition 1. □

As we pointed out in Section 1.1, the monotonicity of entailment contravenes, in a sense, the semantic entrenchment idea, since it allows semantically irrelevant wffs to occur among premises. In the case of strong sc-entailment, however, the difficulty is solved in a radical way: a strongly sc-entailing set is "minimal" with regard to the transmission of truth and, since no proper superset of a set $X$ that strongly sc-entails a wff $A$ strongly sc-entails $A$ as well, adding an "irrelevant" wff to $X$ results in the lack of strong sc-entailment of $A$ from $X$ enriched in this way.

As a consequence of Proposition 15 we get:

**Corollary 6** *Let $X \neq Y$. If $X \models\!\!\!< A$ and $Y \models\!\!\!< A$, then $X \not\subset Y$ and $Y \not\subset X$.*

Thus any two different sets that strongly sc-entail a given wff $A$ are not included in one another. It happens, however, that such such sets are not disjoint.

By the clause 3 of Theorem 5, each proper subset of a strongly sc-entailing set is consistent. Strong sc- and mc-entailment do not differ in this respect. As we have seen, however, there exist strongly mc-entailing sets which are inconsistent. According to the clause 1 of Theorem 5, this never happens in the case of strong sc-entailment. Anyway, strong sc-entailment is free of the drawback (I) pointed out in Section 1.1. Let us add: free, again, in a radical way, since inconsistent sets do not strongly sc-entail any wffs. As an immediate consequence of Theorem 5 one gets:

**Proposition 16** *No wff is strongly sc-entailed by an inconsistent set.*

Observe that the following holds as well:

**Proposition 17** *There is no set of wffs that strongly sc-entails an inconsistent wff.*

**Proof** By Proposition 4. □

Thus inconsistencies are outside the realm of strong sc-entailment: no inconsistent set belongs to the domain of $\models\!\!\!<$ and no inconsistent wff belongs to the range of the relation. No doubt, a paraconsistent logician would dislike strong sc-entailment, although $\models\!\!\!<$ *is*, technically but trivially, paraconsistent.[11]

The case of validities is more complicated. By Proposition 6 we get:

**Proposition 18** *If $X \models\!\!\!< A$, then no wff in $X$ is valid.*

Proposition 7 yields:

**Proposition 19** *Let $C$ be a valid wff. Then $X \models\!\!\!< C$ iff $X = \emptyset$.*

Therefore strongly mc-entailing sets do not contain valid wffs, and a valid wff is strongly mc-entailed only by the empty set. If follows that strong sc-entailment is free of the drawback (II) pointed out in Section 1.1.

One can prove that valid wffs are exactly these wffs which are strongly sc-entailed by the empty set.

**Proposition 20** *A wff $C$ is valid iff $\emptyset \models\!\!\!< C$.*

**Proof** Let $C$ be a valid wff. Thus $\emptyset \Vert\!\!= \{C\}$. Clearly, $\emptyset \Vert\!\!\not= \emptyset$. When $X = \emptyset$, the clause 2 of Definition 6 is trivially fulfilled. Hence $\emptyset \models\!\!\!< C$.

Assume that $\emptyset \models\!\!\!< C$. Thus $\emptyset \models C$ and hence $C$ is valid. □

The claim of Proposition 20 is analogous to the respective claim for sc-entailment traditionally construed: valid wffs are exactly the wffs entailed by the empty set. The difference between strong sc-entailment and sc-entailment simpliciter lies in the fact that valid wffs are strongly mc-entailed *only* by the empty set (cf. Proposition 19).

As an immediate consequence of Proposition 10 we get:

**Proposition 21** $\{A\} \Vdash\!\!< A$ *iff* *A is neither valid nor inconsistent.*

Thus a wff is strongly sc-entailed by itself just in case the wff is neither valid nor inconsistent.

It is worth emphasizing that the following holds:

**Proposition 22** *If X has at least two elements and $A \in X$, then $X \not\Vdash\!\!< A$.*

**Proof** Suppose otherwise. Then, by Theorem 1, there is no proper subset $Z$ of $X$ such that $Z \Vert\!\!= \{A\}$. Therefore $\{A\} \Vert\!\!\not= \{A\}$, which is impossible. □

Hence, with the exception of singleton sets (cf. Proposition 21), strongly sc-entailed wffs do not belong to the sets that strongly sc-entail them.

**Proposition 23** *Let A be a consistent wff and B be a non-valid wff. Then $\{A\} \Vdash\!\!< B$ iff $\{A\} \models B$.*

**Proof** It suffices to observe that $A$ is consistent iff $\{A\} \Vert\!\!\not= \emptyset$, and that $B$ is not valid iff $\emptyset \Vert\!\!\not= \{B\}$. □

Thus, as long as $A$ is consistent and $B$ is not valid, strong sc-entailment of $B$ from $A$ and (traditionally construed) entailment of $B$ from $A$ coincide. It follows that once we have a thesis of CPL of the form $A \to B$ such that $A$ is consistent and $B$ is not valid, we know that the antecedent[12], $A$, strongly sc-entails the succedent, $B$. But one *cannot* generalize Proposition 23 to the case of non-singleton sets of premises. For instance, although we have:

$$\{p \wedge q\} \Vdash\!\!< p \tag{24}$$

$$\{p \wedge q\} \Vdash\!\!< p \vee q \tag{25}$$

we *do not* have:

$$\{p, q\} \Vdash\!\!< p$$
$$\{p, q\} \Vdash\!\!< p \vee q$$

As for strong sc-entailment, to rely on premises only listed need not equal relying on their conjunction. It happens that a conjunction strongly sc-entails more than the mere set of conjuncts. What about the transition from a set of premises to their conjunction? We are not free in making the transition. There are important conditions to be met:

**Proposition 24** *$\{A, B\} \Vdash\!\!< A \wedge B$ iff the set $\{A, B\}$ is consistent and $A \not\Leftrightarrow A \wedge B$ as well as $B \not\Leftrightarrow A \wedge B$.*

**Proposition 25** *$\{A_1, \ldots, A_n\} \Vdash\!\!< A_1 \wedge \ldots \wedge A_n$ iff the set $\{A_1, \ldots, A_n\}$ is consistent and $A_1 \wedge \ldots A_{j-1} \wedge A_{j+1} \wedge \ldots \wedge A_n \not\Leftrightarrow A_1 \wedge \ldots \wedge A_n$ for $j = 1, \ldots, n$.*

Of course, once a transition from a set of of premises to their conjunction is legitimate, we can rely on the conjunction and apply:

**Proposition 26** *Let $\{A_1, \ldots, A_n\}$ be a consistent set of wffs and B be a non-valid wff. Then $\{A_1 \wedge \ldots \wedge A_n\} \Vdash\!\!< B$ iff $\{A_1, \ldots, A_n\} \models B$.*

**Proof** By Proposition 23. It suffices to observe that $\{A_1, \ldots, A_n\}$ is consistent iff $A_1 \wedge \ldots \wedge A_n$ is consistent. □

**4.3 Strong Sc-entailment versus Strong Mc-entailment** In the case of CPL, mc-entailment of a finite set can be identified with sc-entailment of a disjunction of all the elements of the set, i.e. if $Y$ is a finite set, then $X \Vdash Y$ iff $X \models \bigvee Y$. However, even in the case of CPL, strong mc-entailment and strong sc-entailment are not linked in this way. For instance, we have:

$$\{p\} \models\!\!\prec p \vee q \tag{26}$$

but we *do not* have:

$$\{p\} \Vdash\!\!\prec \{p, q\} \tag{27}$$

(27) does not hold because $\{p\} \Vdash (\{p,q\} \setminus \{q\})$.

The above observation can be generalized. Consider a wff of the form $A \vee B$. According to Theorem 4, $X$ strongly sc-entails $A \vee B$ iff the following condition is met:

$$X \cup \{\overline{A \vee B}\} \textit{ is a } \mathsf{MI}\textit{-set and } \overline{A \vee B} \notin X \tag{28}$$

By Theorem 2, $X$ strongly mc-entails $\{A, B\}$ iff it holds that

$$(X \cap \{\overline{A}, \overline{B}\}) = \emptyset \textit{ and } X \cup \{\overline{A}, \overline{B}\} \textit{ is a } \mathsf{MI}\textit{-set} \tag{29}$$

(28) and (29) are not equivalent: (29) implies (28), but not the other way round. The following is true:

**Corollary 7** *If $X$ strongly mc-entails a non-empty set of wffs $Y$, then $X$ strongly sc-entails a disjunction of all the wffs in $Y$. It happens, however, that a disjunction of all the elements of $Y$ is strongly sc-entailed by $X$, but $X$ does not strongly mc-entail $Y$.*

The non-emptiness assumption is unavoidable.[13]

When standard sc- and mc-entailments are considered, it is possible that a set $X$ mc-entails a non-singleton set $Y$ and, at the same time, sc-entails some wff(s) in $Y$. This, however, never happens to strong mc- and sc-entailments.

**Proposition 27** *If $X \Vdash\!\!\prec Y$ and $Y$ has at least two elements, then $X \not\models\!\!\prec B$ for every $B \in Y$.*

**Proof** By Theorem 1 and Definition 8. □

As a consequence of Definition 8 and Proposition 27 we get:

**Corollary 8** *Let $X \Vdash\!\!\prec Y$ and $Y \neq \emptyset$. Then $X$ strongly sc-entails an element of $Y$ only if $Y$ is a singleton set; otherwise no wff in $Y$ is strongly sc-entailed by $X$.*

Finally, let us prove:

**Proposition 28** $X \models\!\!\prec A$ *iff* $\{\overline{A}\} \Vdash\!\!\prec \overline{X}$.

**Proof** If $X \models\!\!\prec A$, then, by Theorem 5, the set $X$ is consistent and $X \models A$. Therefore $\emptyset \not\Vdash \overline{X}$ and $\{\overline{A}\} \Vdash \overline{X}$. Suppose that $\{\overline{A}\} \Vdash Z$ for some proper subset $Z$ of $\overline{X}$. It follows that $\overline{Z} \models A$. But $\overline{Z}$ is a proper subset of $X$. Thus $X \not\models\!\!\prec A$ by Theorem 5. A contradiction. Hence $\{\overline{A}\}$ does not mc-entail any proper subset of $\overline{X}$. Therefore $\{\overline{A}\} \Vdash\!\!\prec \overline{X}$.

If $\{\overline{A}\} \Vdash\!\!\prec \overline{X}$, then, by Proposition 11, $\{\overline{A}\} \cup X$ is a MI-set and $\{\overline{A}\} \cap X = \emptyset$. Hence $\overline{A} \notin X$. Therefore, by Theorem 4, $X \models\!\!\prec A$. □

**4.4 Strict Finiteness and Entrenchment** Since strong sc-entailment is an instance of strong mc-entailment, it is strictly finitistic w.r.t. the entailing sets.

**Theorem 6** *If $X \Vdash A$, then $X$ is a finite set.*

**Proof** By Theorem 3 and Definition 8. □

When we are dealing with sc-entailment traditionally construed, the compactness claim (♣)[14] only yields that a wff sc-entailed by an infinite set of wffs is also sc-entailed by some finite subset(s) of the set. The case of strong sc-entailment is different: no wff is strongly sc-entailed by an infinite set of wffs (recall that strong sc-entailment is not monotone).

Yet, as long as consistent sets are considered, nothing is lost: a wff sc-entailed by a consistent set of wffs, either finite or infinite, is also strongly sc-entailed by some (finite) subset of the set. This is due to:

**Theorem 7** *If $X \models A$ and $X$ is consistent, then there exists at least one finite subset $Y$ of $X$ such that $Y \Vdash A$.*

**Proof** If $X \models A$ and $X$ is consistent, then $A$ is a consistent wff.

Assume that $A$ is valid. Then $\emptyset \Vdash A$ by Proposition 19.

Now assume that $A$ is not valid. Since $X \models A$, there exists a non-empty and finite subset of $X$, say, $X^*$, such that $X^* \models A$. Clearly, $X^*$ is consistent, because $X$ is consistent. Observe that $\overline{A} \notin X^*$; otherwise $X^*$ would have been inconsistent. Since $A$ is not valid, $\overline{A}$ is consistent.

Suppose that $X^*$ has $n$ elements, where $n \geq 1$.

If $n = 1$, $A$ is sc-entailed by the corresponding singleton set $X^*$ included in $X$. The set is consistent. Since $\emptyset$ is the only proper subset of $X^*$ and $A$ is not valid, we finally get $X^* \Vdash A$.

Let $n > 1$. We define the following families of sets:

$$\Theta_1 = \{Z : Z = \{B, \overline{A}\},\ where\ B \in X^*\} \tag{30}$$

$$\Theta_n = \{X^* \cup \{\overline{A}\}\} \tag{31}$$

For each $j$, where $1 < j < n$, we define the corresponding family of sets $\Theta_j$ as follows:

$$\Theta_j = \{Z : Z = \{B_1, \ldots, B_j, \overline{A}\},\ where\ \{B_1, \ldots, B_j\} \subset X^*\} \tag{32}$$

Note that each element of $\Theta_k$, where $1 \leq k \leq n$, can be displayed as:

$$Z' \cup \{\overline{A}\} \tag{33}$$

where $Z'$ is a $k$-element subset of $X^*$.

We consider the following sequence of families of sets of wffs:

$$\Theta_1, \Theta_2, \ldots, \Theta_n \tag{34}$$

Suppose that each term of (34) has only consistent sets of wffs as the elements. Since $\Theta_n = \{X^* \cup \{\overline{A}\}\}$, it follows that the set $X^* \cup \{\overline{A}\}$ is consistent, that is, $X^* \not\models A$. A contradiction. Hence some term(s) of (34) have inconsistent sets of wffs as elements.

There are two possibilities: (i) every term of the sequence (34) has an inconsistent element, or (ii) only some, but not all terms of (34) have inconsistent elements.

If (i) holds, then at least one element of $\Theta_1$ is inconsistent. Let $Z_1$ be an inconsistent element of $\Theta_1$. The set $Z_1$ can be displayed as:

$$\{B, \overline{A}\} \tag{35}$$

where $B \in X^*$. Both $B$ and $\overline{A}$ are consistent wffs and hence $\{B\}$ and $\{\overline{A}\}$ are consistent sets. Hence $Z_1$ is a MI-set. But $\overline{A} \notin X^*$ and thus $B \neq \overline{A}$. Therefore, by Theorem 4, $\{B\} \models\!\!\!\!< A$. Clearly, $\{B\}$ is a finite subset of $X$.

If (ii) is the case, then there exists the least index, $k$, such that at least one element of $\Theta_k$ is inconsistent and $\Theta_{k-1}$ comprises only consistent sets of wffs. Observe that $k > 1$; for if $k = 1$, each term of (34) has some inconsistent element(s). Let $Z_k$ be an inconsistent element of $\Theta_k$. $Z_k$ is of the form:

$$\{B_1, \ldots, B_k, \overline{A}\} \tag{36}$$

where $\{B_1, \ldots, B_k\}$ is a $k$-element subset of $X^*$. Hence $\{B_1, \ldots, B_k\}$ is consistent. Recall that $\overline{A} \notin \{B_1, \ldots, B_k\}$. Observe that for each $B \in \{B_1, \ldots, B_k\}$, the set $(Z_k \setminus \{B\})$ belongs to $\Theta_{k-1}$ and thus is consistent. Thus $Z_k$ is a MI-set. Therefore $\{B_1, \ldots, B_k\} \models\!\!\!\!< A$ by Theorem 4. Needless to say, $\{B_1, \ldots, B_k\}$ is a finite subset of $X$. □

The intuitive content of Theorem 7 is this: sc-entailment from a given, finite or infinite, consistent set of wffs boils down to strong sc-entailment from, one or more, finite subset(s) of the set.

**4.5 An Application: Contraction** Let us imagine that we are working with a non-empty and consistent set of CPL-wffs $X$ (for instance, representing a database or a belief base) and that a CPL-wff $A$ has been derived from $X$, where $A$ is not CPL-valid. We assume that the derivation mechanism used preserves CPL-entailment. Let $s$ be the derivation which has been actually performed. Now suppose that we have strong, though independent from $X$, reasons to believe that $\neg A$ rather than $A$ is the case. In this situation $X$ should be "contracted" in a way that prevents the appearance of $A$ as a conclusion of any legitimate (i.e. preserving CPL-entailment) derivation from the contracted set. Since $s$ is the derivation that has been performed, one can examine $s$ in order to identify the elements of $X$ used as premises in $s$, and then contract $X$ by removing from it at least one wff which was used as a premise in $s$. This, however, will not do: it is possible that $A$ is CPL-entailed by many subsets of $X$, including some that do not contain the just removed wff, and thus $A$ can still be legitimately derived from the set contracted in the above manner. Examining all the possible but legitimate derivations of $A$ from $X$ constitutes a difficult if not a hopeless task. Yet, a solution is suggested by the content of Theorem 7. By and large, it suffices to consider all the finite subsets of $X$ that strongly sc-entail $A$, and to remove from $X$ exactly one element of every such set. A contracted set obtained in this way does not CPL-entail the wff $A$ and therefore no legitimate derivation leads from the set to $A$. This is due to the following:

**Theorem 8** *Let $X$ be a non-empty and consistent set of wffs, and let $A$ be a non-valid wff such that $X \models A$. Let $\Xi = \{W \subseteq X : W \models\!\!\!\!< A\}$, and let $Z$ be a set of wffs such that for each $X' \in \Xi$, the set $Z$ contains exactly one element of $X'$, and each element of $Z$ belongs to some set $X' \in \Xi$. Then $(X \setminus Z) \not\models A$.*

**Proof** By Theorem 7, the family $\Xi$ is non-empty. By assumption, $A$ is not valid. Thus, by Proposition 20, $X' \neq \emptyset$ for each $X' \in \Xi$, and hence $Z \neq \emptyset$.

The set $(X \setminus Z)$ is consistent, since $X$ is, by assumption, consistent.

Suppose that $(X \setminus Z) \models A$. It follows that $(X \setminus Z) \neq \emptyset$ (as $A$ is not valid) and, by Theorem 7, that $Y \models\!\!\!\!< A$ for some finite subset $Y$ of $(X \setminus Z)$. Moreover, $Y \neq \emptyset$;

otherwise $A$ would have been valid. But the only subsets of $X$ that strongly sc-entail $A$ are the sets in $\Xi$. Hence $Y = X^\circ$ for some element, $X^\circ$, of $\Xi$. But $(X' \cap Z) \neq \emptyset$ for each $X' \in \Xi$ and hence $(X^\circ \cap Z) \neq \emptyset$. On the other hand, $(Y \cap Z) = \emptyset$ due to the fact that $Y$ is a subset of $(X \setminus Z)$. It follows that $Y \neq X^\circ$. We arrive at a contradiction. Therefore $(X \setminus Z) \not\models A$. □

**Remark 3** If $X$ is finite, then the corresponding $\Xi$ is finite as well. A procedure of identifying all the subsets of a finite set entailing a wff that strongly sc-entail the wff is sketched in Appendix 1.

## 5 Towards a Proof-theoretic Account of Strong Entailments

As Theorem 2 shows, a problem of the form:

(**P**) *Does $X$ strongly mc-entail $Y$?*

splits into two sub-problems:

**(P$_1$):** *Is it the case that $X \cap \overline{Y} = \emptyset$?*
**(P$_2$):** *Is $X \cup \overline{Y}$ a* MI-*set?*

Similarly, due to Theorem 4, a problem of the form:

(**P'**) *Does $X$ strongly sc-entail $A$?*

splits into:

**(P'$_1$):** *Is it the case that $\overline{A} \notin X$?*
**(P'$_2$):** *Is $X \cup \{\overline{A}\}$ a* MI-*set?*

**P**$_1$ and **P'**$_1$ are syntactic issues which can be resolved by simple syntactic means. But either **P**$_2$ or **P'**$_2$ is a problem that pertains to a semantic property. In order to solve it syntactically one needs a proof-theoretic account of MI-sets.

In this section we show that a proof-theoretic account of MI-sets is offered, though indirectly, by some subsystem of the system PMC presented in [11]. By and large, PMC is a sequent-style axiomatic system for *proper multiple-conclusion entailment* (proper mc-entailment for short) in CPL. The relevant subsystem of PMC is a sequent-style axiomatic system for proper mc-entailment from the empty set.

**5.1 Proper Mc-entailment and MI-sets** Proper mc-entailment, $|\!\!\lhd$, is defined by (cf. [11]):

**Definition 9 (Proper mc-entailment)** *Let $Y \neq \emptyset$. $X \;|\!\!\lhd\; Y$ iff $X \;||\!\!= Y$ and for each $B \in Y : X \not\models B$.*

Thus $X$ properly mc-entails $Y$ just in case $X$ mc-entails $Y$, but $X$ does not sc-entail any element of $Y$.

**Example 1** The following hold:

$$\{p \vee q\} \;|\!\!\lhd\; \{p, q\} \tag{37}$$

$$\{p \to q \vee r, p\} \;|\!\!\lhd\; \{q, r\} \tag{38}$$

$$\{p \wedge q \to r, \neg r\} \;|\!\!\lhd\; \{\neg p, \neg q\} \tag{39}$$

$$\emptyset \;|\!\!\lhd\; \{p, \neg p\} \tag{40}$$

Proper mc-entailment and MI-sets are mutually connected. Recall that each MI-set is non-empty. There are two separate cases to be considered: (a) singleton MI-sets, and (b) at least two-element MI-sets. Since we are dealing here with CPL, the non-singleton sets to be considered are finite.

***Notation***. Let $Z = \{C_1, \ldots, C_n\}$, where $n > 1$. By $\curlyvee_{\setminus} Z$ we designate the set:

$$\{C_2 \vee C_3 \vee \ldots \vee C_n, \ldots, C_1 \vee \ldots \vee C_{j-1} \vee C_{j+1} \vee \ldots \vee C_n, C_1 \vee \ldots \vee C_{n-1}\}$$

As $C_1, \ldots, C_n$ are pairwise syntactically distinct, so are the wffs in $\curlyvee_{\setminus}\{C_1, \ldots, C_n\}$.

**Example 2**

$$\curlyvee_{\setminus}\{p, q\} = \{p, q\}$$
$$\curlyvee_{\setminus}\{p, q, r\} = \{q \vee r, p \vee r, p \vee q\}$$
$$\curlyvee_{\setminus}\{p, q, r, s\} = \{q \vee r \vee s, p \vee r \vee s, p \vee q \vee s, p \vee q \vee r\}$$

Observe that the following holds:

**Lemma 2** *Let $n > 1$. $\emptyset \Vert\!\!\models \{C_1, \ldots, C_n\}$ iff $\emptyset \Vert\!\!\models \curlyvee_{\setminus}\{C_1, \ldots, C_n\}$.*

Let us now prove:

**Theorem 9** *Let $n > 1$. $\{C_1, \ldots, C_n\}$ is a MI-set iff $\emptyset \Vert\!\!\lhd \curlyvee_{\setminus}\{\overline{C_1}, \ldots, \overline{C_n}\}$.*

**Proof**

($\Rightarrow$) If $\{C_1, \ldots, C_n\}$ is a MI-set, then, by Proposition 14, $\emptyset \Vert\!\!\prec \{\overline{C_1}, \ldots, \overline{C_n}\}$. Hence, by Lemma 2, $\emptyset \Vert\!\!\models \curlyvee_{\setminus}\{\overline{C_1}, \ldots, \overline{C_n}\}$. If $\{C_1, \ldots, C_n\}$ is a MI-set, then, by Corollary 1, for each $j$, where $1 \leq j \leq n$, the set $(\{C_1, \ldots, C_n\} \setminus \{C_j\})$ is consistent. Thus no element of $\curlyvee_{\setminus}\{\overline{C_1}, \ldots, \overline{C_n}\}$ is valid. Hence $\emptyset \not\models B$ for each $B \in \curlyvee_{\setminus}\{\overline{C_1}, \ldots, \overline{C_n}\}$. Therefore $\emptyset \Vert\!\!\lhd \curlyvee_{\setminus}\{\overline{C_1}, \ldots, \overline{C_n}\}$.

($\Leftarrow$) Assume that $\emptyset \Vert\!\!\lhd \curlyvee_{\setminus}\{\overline{C_1}, \ldots, \overline{C_n}\}$. Hence $\emptyset \Vert\!\!\models \curlyvee_{\setminus}\{\overline{C_1}, \ldots, \overline{C_n}\}$ and thus, by Lemma 2, $\emptyset \Vert\!\!\models \{\overline{C_1}, \ldots, \overline{C_n}\}$. It follows that the set $\{C_1, \ldots, C_n\}$ is inconsistent. At the same time we have $\emptyset \not\models B$ for each $B \in \curlyvee_{\setminus}\{\overline{C_1}, \ldots, \overline{C_n}\}$. Thus each proper subset of $\{C_1, \ldots, C_n\}$ is consistent. Therefore $\{C_1, \ldots, C_n\}$ is a MI-set. □

For singleton sets we have:

**Theorem 10** *$\{C\}$ is a MI-set iff $\emptyset \Vert\!\!\lhd \{\overline{C} \wedge p, \overline{C} \wedge \neg p\}$.*

**Proof**

($\Rightarrow$) If $\{C\}$ is a MI-set, then $C$ is an inconsistent wff and hence $\overline{C}$ is a valid wff. Thus we have:

$$\emptyset \Vert\!\!\models \{\overline{C} \wedge p, \overline{C} \wedge \neg p\} \tag{41}$$

On the other hand, it is clear that the following hold as well:

$$\emptyset \not\models \overline{C} \wedge p \tag{42}$$

$$\emptyset \not\models \overline{C} \wedge \neg p \tag{43}$$

Therefore $\emptyset \Vert\!\!\lhd \{\overline{C} \wedge p, \overline{C} \wedge \neg p\}$.

($\Leftarrow$) If $\emptyset \Vert\!\!\lhd \{\overline{C} \wedge p, \overline{C} \wedge \neg p\}$ is the case, then (41) is true and hence the following holds as well:

$$\emptyset \models \overline{C} \wedge (p \vee \neg p) \tag{44}$$

Thus we have:

$$\emptyset \models \overline{C} \tag{45}$$

It follows that $C$ is an inconsistent wff. Therefore $\{C\}$ is a MI-set. □

**5.2 MI-sets and Proper Mc-entailment From the Empty Set: the System $\mathsf{PMC}_{\emptyset}$** As theorems 9 and 10 show, the property of being a MI-set is, strictly speaking, connected with proper mc-entailment (of the "associated" sets, i.e. sets of the form $\curlyvee_{\setminus} Z$ or $\{\overline{C} \wedge p, \overline{C} \wedge \neg p\}$) from the empty set. This suggests that a proof-theoretic account of strong entailments can be based on a system weaker than PMC, namely on a system characterizing proper mc-entailment from the empty set only. In what follows we describe such a system. We label it as $\mathsf{PMC}_{\emptyset}$. The idea (but not the label) comes from the first author of [11], who developed the system at the beginning, but then the joint work resulted in PMC.[15]

We operate with sequents with empty antecedents and non-empty but finite succedents. To be more precise, by a *sequent* we will mean an expression of the form $\vdash Y$, where $Y$ is a non-empty and finite set of CPL-wffs. A sequent $\vdash Y$ is in the *normal form* iff each $B \in Y$ is in the conjunctive normal form, i.e. is an elementary disjunction[16] or a conjunction of elementary disjunctions.

The inscription "$A \in$ CPL" means: "$A$ is a thesis of CPL." Since we are dealing with finite sets, we characterize sets by listing their elements; for brevity, we omit curly brackets. As usual, we write $X, A$ for $X \cup \{A\}$, and $X, Y$ for $X \cup Y$.

An *axiom* of $\mathsf{PMC}_{\emptyset}$ is a sequent falling under the schema:

$$\vdash Y \tag{46}$$

such that each element of $Y$ is an elementary disjunction, $\bigvee Y \in$ CPL, and $B \notin$ CPL for each $B \in Y$. Here are examples of axioms of $\mathsf{PMC}_{\emptyset}$:

$$\vdash p, \neg p \tag{47}$$

$$\vdash p \vee \neg q, q \vee \neg p \tag{48}$$

$$\vdash p \vee q, \neg p, \neg q \tag{49}$$

There are only two (primary) rules of $\mathsf{PMC}_{\emptyset}$, namely:

R'$_1$: $$\frac{\vdash Y, A \qquad \vdash Y, B}{\vdash Y, A \wedge B}$$

R'$_2$: $$\frac{\vdash Y, A}{\vdash Y, B} \quad \text{where } (A \leftrightarrow B) \in \mathsf{CPL}$$

A *proof* of a sequent $\vdash Y$ in $\mathsf{PMC}_{\emptyset}$ is a finite labeled tree regulated by the rules of $\mathsf{PMC}_{\emptyset}$ where the leaves are labeled with axioms and $\vdash Y$ labels the root. A sequent $\vdash Y$ is *provable* in $\mathsf{PMC}_{\emptyset}$ iff $\vdash Y$ has at least one proof in $\mathsf{PMC}_{\emptyset}$.

One can easily show that the following holds:

**Lemma 3** *If the sequent $\vdash Y$ is provable in $\mathsf{PMC}_{\emptyset}$, then $\emptyset \Vert\!\!\lhd Y$.*

The following holds as well:

**Lemma 4** *Let $\vdash Y$ be a sequent in the normal form. If $\emptyset \Vert\!\!\lhd Y$, then $\vdash Y$ is provable in $\mathsf{PMC}_{\emptyset}$.*

The proof of Lemma 4 is analogous to the proof of Lemma 2 in [11]; in order to make this paper self-contained we present it in Appendix 2. Since for each wff $A$ there exists a wff $B$ in the conjunctive normal form such that $(A \leftrightarrow B) \in$ CPL, and we have rule R'$_2$, by Lemma 4 we get:

**Lemma 5** *If $\emptyset \Vert\!\!\lhd Y$, then the sequent $\vdash Y$ is provable in $\mathsf{PMC}_{\emptyset}$.*

Lemma 3 together with Lemma 5 yield:

**Theorem 11** *Let $Y$ be a non-empty and finite set of* CPL-*wffs.* $\emptyset \Vert\!\!\prec Y$ *iff the sequent* $\vdash Y$ *is provable in* $\mathsf{PMC}_{\emptyset}$.

Each MI-set of CPL-wffs is finite and non-empty. By theorems 11, 9 and 10 we finally get:

**Theorem 12**

1. *Let $n > 1$. $\{C_1, \ldots, C_n\}$ is a* MI-*set iff the sequent:*

$$\vdash \curlyvee_{\setminus}\{\overline{C_1}, \ldots, \overline{C_n}\}$$

   *is provable in* $\mathsf{PMC}_{\emptyset}$.

2. *$\{C\}$ is a* MI-*set iff the sequent:*

$$\vdash \overline{C} \wedge p, \overline{C} \wedge \neg p$$

   *is provable in* $\mathsf{PMC}_{\emptyset}$.

Therefore one can rely on $\mathsf{PMC}_{\emptyset}$ proofs in solving problems of being a MI-set and, due to theorems 2 and 4, in showing that strong entailments hold. Let us end with some examples.

**Example 3** Let $X = \{p \to q, p, \neg q\}$. We will show that $X$ is a MI-set by proving the corresponding sequent in $\mathsf{PMC}_{\emptyset}$.

Observe that:

$$\overline{X} = \{\neg(p \to q), \neg p, q\}$$

$$\curlyvee_{\setminus}\overline{X} = \{\neg p \vee q, \neg(p \to q) \vee q, \neg(p \to q) \vee \neg p\}$$

Here is a $\mathsf{PMC}_{\emptyset}$ proof of the sequent $\vdash \curlyvee_{\setminus}\overline{X}$.

$$\vdash \neg p \vee q, p \vee q, \neg q \vee p \qquad \text{(Ax)}$$
$$\vdash \neg p \vee q, (p \vee q) \wedge (\neg q \vee q), \neg q \vee p \qquad \text{(R'}_2\text{)}$$
$$\vdash \neg p \vee q, (p \vee q) \wedge (\neg q \vee q), (p \vee \neg p) \wedge (\neg q \vee p) \qquad \text{(R'}_2\text{)}$$
$$\vdash \neg p \vee q, (p \wedge \neg q) \vee q, (p \vee \neg p) \wedge (\neg q \vee p) \qquad \text{(R'}_2\text{)}$$
$$\vdash \neg p \vee q, (p \wedge \neg q) \vee q, (p \wedge \neg q) \vee \neg p \qquad \text{(R'}_2\text{)}$$
$$\vdash \neg p \vee q, \neg(p \to q) \vee q, (p \wedge \neg q) \vee \neg p \qquad \text{(R'}_2\text{)}$$
$$\vdash \neg p \vee q, \neg(p \to q) \vee q, \neg(p \to q) \vee \neg p \qquad \text{(R'}_2\text{)}$$

**Example 4** Let $Y = \{\neg(p \vee q), \neg(\neg p \wedge \neg q)\}$. Then $\overline{Y} = \{p \vee q, \neg p \wedge \neg q)\}$ and $\curlyvee_{\setminus}\overline{Y} = \{p \vee q, \neg p \wedge \neg q)\}$. Here is a $\mathsf{PMC}_{\emptyset}$ proof of the sequent $\vdash p \vee q, \neg p \wedge \neg q$.

$$\vdash p \vee q, \neg p \quad \text{(Ax)} \qquad \vdash p \vee q, \neg q \quad \text{(Ax)}$$
$$\vdash p \vee q, \neg p \wedge \neg q \qquad \text{(R'}_1\text{)}$$

Therefore $\{\neg(p \vee q), \neg(\neg p \wedge \neg q)\}$ is a MI-set. We have $(\neg p \wedge \neg q) \notin \{\neg(p \vee q)\}$ and $\neg(p \vee q) \notin \{\neg(\neg p \wedge \neg q)\}$. Hence:

$$\{\neg(p \vee q)\} \mid\!\!\prec \neg p \wedge \neg q \tag{50}$$

$$\{\neg(\neg p \wedge \neg q)\} \mid\!\!\prec p \vee q \tag{51}$$

We also get:

$$\{\neg(p \vee q), \neg(\neg p \wedge \neg q)\} \Vert\!\!\prec \emptyset \tag{52}$$

$$\emptyset \Vert\!\!\prec \{p \vee q, \neg p \wedge \neg q\} \tag{53}$$

**Example 5** We want to check if $\{\neg p, \neg q\} \models\!\prec \neg(p \vee q)$ holds. Since $(p \vee q) \notin \{\neg p, \neg q\}$, it suffices to check if $\{\neg p, \neg q, p \vee q\}$ is a MI-set. This, however, holds iff the sequent:

$$\vdash q \vee \neg(p \vee q), p \vee \neg(p \vee q), p \vee q \tag{54}$$

is provable in $\mathsf{PMC}_\emptyset$. Here is a $\mathsf{PMC}_\emptyset$ proof of the sequent (54):

$$\begin{array}{cc}
\vdash q \vee \neg p, p \vee \neg q, p \vee q & (\mathsf{Ax}) \\
\vdash (q \vee \neg p) \wedge (q \vee \neg q), p \vee \neg q, p \vee q & (\mathsf{R'}_2) \\
\vdash q \vee (\neg p \wedge \neg q), p \vee \neg q, p \vee q & (\mathsf{R'}_2) \\
\vdash q \vee \neg(p \vee q), p \vee \neg q, p \vee q & (\mathsf{R'}_2) \\
\vdash q \vee \neg(p \vee q), (p \vee \neg q) \wedge (p \vee \neg p), p \vee q & (\mathsf{R'}_2) \\
\vdash q \vee \neg(p \vee q), p \vee (\neg q \wedge \neg p), p \vee q & (\mathsf{R'}_2) \\
\vdash q \vee \neg(p \vee q), p \vee \neg(p \vee q), p \vee q & (\mathsf{R'}_2)
\end{array}$$

## 6 Final remarks

We already paid some attention to possible applications of the concept of strong sc-entailment (cf. sections 4.1 and 4.5). Let us end this paper with a few remarks about possible applications of the more general concept of strong mc-entailment.

The concept of argument is sometimes generalized to include arguments containing any finite number of conclusions. As a result we get an unproblematic class of arguments having exactly one conclusion – let us call them *sc-arguments* – and a problematic class of arguments having at least two (but still finitely many) conclusions. Let us call the latter *mc-arguments*.

As we observed (see Section 4.5), strong sc-entailment is, in principle, the semantic relation which holds between premises and conclusions of relevant (in the Lehrer's sense) deductive sc-arguments. By analogy, strong mc-entailment can be viewed as singling out a class of *relevant* mc-arguments. To be more precise, one can stipulate that a relevant mc-argument is an mc-argument whose premises strongly mc-entail the *set* of conclusions of the argument. The properties of strong mc-entailment pointed out above seem to speak in favour of this proposal.

There is an ongoing discussion as to whether mc-arguments are artifacts.[17] An mc-argument is often identified with an sc-argument whose conclusion is a disjunction of all the "conclusions" of the respective mc-argument. Without pretending to resolve the issue, let us only notice the following.

Consider:

$$\frac{p}{p \vee q} \tag{55}$$

and

$$\frac{p}{p, q} \tag{56}$$

Since $\{p\} \models\!\prec p \vee q$ holds, (55) constitutes a relevant sc-argument. But $\{p\} \parallel\!\prec \{p, q\}$ does not hold and thus (56) is not a relevant mc-argument.

In general, it happens that, having premises fixed, there exist relevant sc-arguments leading from the premises(s) to a disjunction, but there is no relevant mc-argument leading from the premise(s) to the set of disjuncts.

Now let us consider:

$$\frac{p \vee q}{p, q} \tag{57}$$

and

$$\frac{p \vee q}{p} \tag{58}$$

$$\frac{p \vee q}{q} \tag{59}$$

(57) is a relevant mc-argument, while (58) and (59) are not relevant sc-arguments. This is not an exception, but a rule. Due to Proposition 27, if an mc-argument leading from a disjunction to the disjuncts is relevant, then there is no relevant sc-argument that leads from the disjunction to a disjunct.[18]

The last remark is this. Multiple-conclusion entailment is one of the central conceptual tools of Inferential Erotetic Logic, IEL,[19] and Minimal Erotetic Semantics, MiES.[20] Since mc-entailment based on Classical Logic is monotone, its applications sometimes lead to unwanted results (cf. [4]). A remedy is to switch to a non-classical logic, a move that is permitted both by IEL and by MiES in their current forms. Another option is to operate with an mc-entailment that is not monotone. It is an open problem whether strong mc-entailment analyzed in this paper is the best candidate here.

## Appendix 1: A Procedure of Identifying All Strongly Entailing Subsets of a Finite Set of Premises

We use the notation introduced in the proof of Theorem 7.

Let $X^*$ be a finite consistent set of wffs such that $X^* \models A$. Our aim is to find all the finite subsets of $X^*$ that strongly sc-entail $A$.

If $A$ is a valid wff, $\emptyset$ is the only finite subset of $X^*$ that strongly sc-entails $A$. So the problem is solved in a trivial way.

If $A$ is not valid, $\emptyset$ does not strongly sc-entail $A$. At the same time $X^* \neq \emptyset$, $A$ is consistent and $\overline{A} \notin X^*$. The problem reduces to identifying all the elements of the following set:

$$\bigcup_{i=1}^{i=n} \Theta_i \tag{60}$$

that are MI-sets.[21] (60) is the union of all the terms of the sequence (34) defined in the proof of Theorem 7 and hence each element of (60) is of the form:

$$Z \cup \{\overline{A}\} \tag{61}$$

where $Z \subset X^*$ and $\overline{A} \notin Z$. Thus when we have a MI-set of the form (61), we know that $Z$ is a subset of $X^*$ looked for.

Here is a procedure that produces the required result.

*Step 1.* We consider the elements of $\Theta_1$ one after another in order to check whether a given element is consistent; it is clear that each inconsistent element of $\Theta_1$ is a MI-set of the required kind. The outcome is:

$$\langle \Psi_1, \Gamma_1 \rangle$$

where $\Psi_1$ is the set of all the inconsistent elements of $\Theta_1$ (and hence MI-sets) if there are any, and $\Psi_1 = \emptyset$ otherwise.

Let:

$$\Psi_1^{\nearrow} =_{df} \{Y \in \Theta_2 : W \subset Y \textit{ for some } W \in \Psi_1\}$$

$\Gamma_1$ is defined as follows.

1. *If* $\Psi_1 = \Theta_1$*, then* $\Gamma_1 = \emptyset$.
2. *If* $\Psi_1 \neq \Theta_1$*, then* $\Gamma_1 = (\Theta_2 \setminus \Psi_1^{\nearrow})$.

We stop if $\Gamma_1 = \emptyset$, otherwise we go to Step 2, at which we consider all the elements of $\Gamma_1$ by checking their consistency.

A comment is in order. When $\Psi_1 = \Theta_1$, each set of the form $\{B, \overline{A}\}$, where $B \in X^*$, happens to be a MI-set and hence the set (60) contains no further MI-sets.

If $\Psi_1 \neq \Theta_1$ and $\Psi_1 \neq \emptyset$, at the second step we consider only the elements of $\Theta_2$ which belong to $\Gamma_1$. This is due to the fact that each element of $\Psi_1^{\nearrow}$ is an inconsistent set which has an inconsistent proper subset. So there is no reason to consider the elements of $\Theta_2$ that belong to $\Psi_1^{\nearrow}$, since we know in advance that none of them is a MI-set. So we consider the set $\Theta_2 \setminus \Psi_1^{\nearrow}$ only. If, however, $\Psi_1 = \emptyset$, we have $\Gamma_1 = \Theta_2$, that is, the whole set $\Theta_2$ is taken into consideration at the next step.

Assume that we did not stop at Step $i$, i.e., that $\Gamma_i \neq \emptyset$.

*Step i+1*. We check the consistency of the consecutive elements of the set $\Gamma_i$ obtained at Step $i$. The outcome is:

$$\langle \Psi_{i+1}, \Gamma_{i+1} \rangle$$

where $\Psi_{i+1}$ is the set of all the inconsistent elements of $\Gamma_i$, if there are any, and $\Psi_{i+1} = \emptyset$ otherwise, while $\Gamma_{i+1}$ is defined by the conditions:

1. *If* $\Psi_{i+1} = \Gamma_i$*, then* $\Gamma_{i+1} = \emptyset$.
2. *If* $\Psi_{i+1} \neq \Gamma_i$*, then*

$$\Gamma_{i+1} = (\Theta_{i+2} \setminus \Psi_{i+1}^{\nearrow}).$$

where

$$\Psi_{i+1}^{\nearrow} =_{df} \{Y \in \Theta_{i+2} : W \subset Y \textit{ for some } W \in \bigcup_{j=1}^{j=i+1} \Psi_j\}$$

We stop if $\Gamma_{i+1} = \emptyset$; otherwise we go to Step $n+2$.

Observe that the elements of $\Psi_{i+1}$, if there are any, are MI-sets. To see this, assume that $Y \in \Psi_{i+1}$. Thus $Y$ has been selected from the elements of $\Gamma_i$, that is, $Y \in \Gamma_i$. Suppose that $Y$ has inconsistent proper subset(s). As $Y$ is finite, the number of its inconsistent proper subsets is also finite. So there exists an index $e < i$ such that some inconsistent proper subset of $Y$, say, $Y^*$, has exactly $e$ elements and no inconsistent proper subset of $Y$ has less than $e$ elements. But each proper subset of $Y^*$ has at most $e-1$ elements and is a proper subset of $Y$. Hence $Y^*$ is a MI-set included in $Y$. As such, $Y^*$ belongs to $\Psi_{e+1}$. It follows that:

$$Y \in \Psi_{e+1}^{\nearrow}$$

and hence

$$Y \in \bigcup_{j=1}^{j=i} \Psi_j^{\nearrow}$$

Therefore $Y \notin \Gamma_i$. A contradiction. Thus $Y$ has no inconsistent proper subset. But $Y$ is inconsistent, because $Y \in \Psi_{i+1}$. Hence $Y$ is a MI-set.

It is clear that the procedure sketched above terminates in a finite number of steps and produces the required result. The result is:

$$\bigcup_{j=1}^{j=k} \Psi_j \tag{62}$$

where $k$ is the number of performed steps. The procedure checks all the subsets of (60) that might have been MI-sets. The set (62) is non-empty due to Theorem 7. Observe that each success, i.e. finding a MI-set, reduces the initial search space. The procedure is based on a idea different from that used in [7].

## Appendix 2. A Proof of Lemma 4

**Lemma 4** *Let* $\vdash Y$ *be a sequent in the normal form. If* $\emptyset \not\Vdash\!\!\vartriangleleft Y$*, then* $\vdash Y$ *is provable in* $\mathsf{PMC}_{\emptyset}$.

**Proof** By assumption, $\vdash Y$ is a sequent in the normal form, that is, the elements of $Y$ are CPL-wffs in the conjunctive normal form. By $\mathsf{r}(Y)$ we designate the number of occurrences of the conjunction connective, $\wedge$, in $Y$. We proceed by induction on $\mathsf{r}(Y)$.

(1) $\mathsf{r}(Y) = 0$. Assume that $\emptyset \not\Vdash\!\!\vartriangleleft Y$. Then $\vdash Y$ is an axiom of $\mathsf{PMC}_{\emptyset}$, so the sequent is provable in the calculus.

(2) $\mathsf{r}(Y) > 0$. Assume that $\emptyset \not\Vdash\!\!\vartriangleleft Y$, where $Y = \{A_1, \ldots, A_n\}$. Then for some $i$, where $1 \leq i \leq n$, we have $A_i = C_1 \wedge \ldots \wedge C_m$ and $m > 1$. At the same time $\not\models A_i$ and thus for some $j$, where $1 \leq j \leq m$, it holds that $\not\models C_j$. Consider the sets:

$$Y_j = \{A_1, \ldots, A_{i-1}, C_j, A_{i+1}, \ldots, A_n\}$$

$$Y_j' = \{A_1, \ldots, A_{i-1}, \bigwedge\{C_k : k \neq j\}, A_{i+1}, \ldots, A_n\}$$

We have $\mathsf{r}(Y_j) < \mathsf{r}(Y)$ and $\mathsf{r}(Y_j') < \mathsf{r}(Y)$, so by the induction hypothesis:

(a) if $\emptyset \not\Vdash\!\!\vartriangleleft Y_j$ then $\vdash Y_j$ is provable in $\mathsf{PMC}_{\emptyset}$;
(b) if $\emptyset \not\Vdash\!\!\vartriangleleft Y_j'$ then $\vdash Y_j'$ is provable in $\mathsf{PMC}_{\emptyset}$.

But when $\emptyset \not\Vdash\!\!\vartriangleleft Y$ holds, we have both $\Vdash\!\models Y_j$ and $\Vdash\!\models Y_j'$. Yet, it also holds that $\not\models C_j$. Thus $\emptyset \not\Vdash\!\!\vartriangleleft Y_j$ and hence, by (a), $\vdash Y_j$ is provable in $\mathsf{PMC}_{\emptyset}$.
(*Case 1*) $\not\models \bigwedge\{C_k : k \neq j\}$. Then $\emptyset \not\Vdash\!\!\vartriangleleft Y_j'$, so, by (b), $\vdash Y_j'$ is provable in $\mathsf{PMC}_{\emptyset}$. Since we have rules R'$_1$ and R'$_2$, and $\vdash Y_j$ is provable as well, it follows that $\vdash Y$ is provable in the calculus.
(*Case 2*) $\models \bigwedge\{C_k : k \neq j\}$. Then $(A_j \leftrightarrow C_j) \in \mathsf{CPL}$, so, by R'$_2$, $\vdash Y$ is provable in $\mathsf{PMC}_{\emptyset}$. □

## Notes

1. The latter statement can be explicated as: "If the truth-conditions of all the premises are met, an entailed conclusion is true as well."

2. It is sometimes claimed that the concept of mc-entailment originates from Gentzen [5] due to his introduction of sequents with sequences of wffs in the succedents. The semantic concept of mc-entailment was explicitly introduced by Carnap (cf. [3]) under the heading 'involution.' Its syntactic counterpart, mc-consequence, was incorporated

into the general theory of logical calculi by Dana Scott [9]. The first monograph devoted to mc-consequence and related concepts (multiple-conclusion calculus, multiple-conclusion rules, etc.) was Shoesmith & Smiley [10].

3. Philosophers tend to think that mc-entailment between sets $X$ and $Y$ is nothing more than sc-entailment of a disjunction of all the formulas in $Y$ from a conjunction of all the formulas in $X$. However, this is not always so; whether this is right depends both on the richness of syntax (the presence/lack of infinite disjunctions and conjunctions) and semantics (the classical/nonclassical meanings of disjunction and conjunction). For details and counterexamples see [11].

4. This concept has found natural applications is many areas, from philosophy of science (cf. e.g. [6]) to theoretical computer science (see e.g. [7]). Minimally inconsistent sets are also called *minimal unsatisfiable (sub)sets* or *unsatisfiable cores*.

5. For example, in a logic that validates the $\omega$-rule, a set of the form $\{\exists x Px\} \cup \{\neg Pa : a \in \mathsf{T}\}$, where $P$ is a predicate and $\mathsf{T}$ is a (countably infinite) set of all closed terms of the language, is an infinite MI-set.

6. As for (17), $\{q\} \setminus \{q\} = \emptyset$, but we have $\{p, \neg p\} \Vdash \emptyset$. In the case of (18) we have $\emptyset \Vdash \{p \vee \neg p\}$.

7. If any, since there are inconsistent sets that do not strongly mc-entail even the empty set. For instance, the set $\{p \wedge \neg p, p\}$ does not strongly entail the empty set because we still have $\{p \wedge \neg p\} \Vdash \emptyset$.

8. Recall that $\{\neg q\} \models \neg p \vee (p \wedge \neg q)$ and hence $\{\neg q\} \Vdash \{\neg p, p \wedge \neg q\}$. On the other hand, $\emptyset \nVdash \{\neg p, p \wedge \neg q\}$, and $\{\neg q\} \nVdash \{\neg p\}$ as well as $\{\neg q\} \nVdash \{p \wedge \neg q\}$.

9. In particular, these and closely related concepts play important roles in adaptive logics; cf. e.g. [1].

10. Observe that (23) equals $\{p, p \rightarrow \neg q, q\}$, since $\overline{\neg q} = q$.

11. It is not the case that each pair $\langle X, A \rangle$ such that $X$ is inconsistent and $A$ is an arbitrary wff belongs to $\mid\!\prec$. This holds because there is *no* pair $\langle X, A \rangle$ with $X$ inconsistent that belongs to $\mid\!\prec$.

12. To be more precise, the singleton set which contains the antecedent.

13. The following example illustrates this. Recall that $\bigvee \emptyset = \bot$. We have $\{p, \neg p\} \,\|\!\!\prec \emptyset$, and, by Proposition 16, we also have $\{p, \neg p\} \not\mid\!\prec \bot$.

14. See page 5.

15. Private communication.

16. That is, a literal (a propositional variable or the negation of a propositional variable) or a disjunction of literals.

17. Cf. [8], [12], [2].

18. By propositions 1 and 27, the general picture is this: (i) if there exists a relevant sc-argument that leads from premises $A_1, \ldots, A_n$ to a conclusion $C$, then there is no relevant mc-argument based on the premises $A_1, \ldots, A_n$ that has $C$ as one of the conclusions; (ii) if there exists a relevant mc-argument based on premises $A_1, \ldots, A_n$ that has $C$ among conclusions, then there is no relevant sc-argument that leads from $A_1, \ldots, A_n$ to $C$.

    Interestingly enough, Proposition 28 enriches the picture with (iii): an sc-argument leading from $A_1, \ldots, A_n$, where $n > 1$, to $C$ is relevant if and only if the single-premise mc-argument that leads from $\neg C$ to $\neg A_1, \ldots, \neg A_n$ is relevant.

19. Cf. e.g. [13], [14].

20. Cf. [15], or [14], chapters 2 and 3. By and large, IEL is a logic of questions that analyses inferences leading to questions and gives an account of validity of these inferences.

21. Algorithms for solving problems of this kind are known; see [7].

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