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REDUCIBILITY OF QUESTIONS TO SETS OF YES-NO QUESTIONS

1. Introductory remarks

The problem of reducibility of questions has many aspects. First, one can speak of reducibility of questions to expressions of some other kind: declarative sentences, imperatives, epistemic imperatives, alethic modalities, etc. Second, we can speak of reducibility of questions of one kind to questions of another kind. The relevant concept of reducibility, however, may be understood in two different ways: as a reducibility of a (single) question of some kind to a (single) question of another kind, or as a reducibility of a (single) question of some kind to a *set* of questions of some kind or kinds. These concepts do not coincide. Although erotetic logicians paid more attention to the first concept of reducibility, the second concept seems not to be of less importance: it often happens that we try to answer a given question by "reducing" it to a number of auxiliary questions.

This paper introduces the concept: *a question is reducible to a non-empty set of questions.* Roughly, we define the concept of reducibility of a question Q to a non-empty set of questions Φ in such a way that the following conditions are fulfilled: (a) if Q is sound (i.e. has a true direct answer), then each question in Φ is also sound; (b) Q can be answered by answering the questions of Φ, and (c) no question in Φ is more complex than Q. We prove that almost all questions may be reduced (in the sense of the proposed definition) to sets made up of yes-no questions, either simple or conditional.

2. The logical basis

(**A**) Let $\mathcal{L}$ be a first-order language enriched with some expressions which enable us to form questions. The *declarative well-formed formulae* (d-wffs for short) of $\mathcal{L}$ are defined as usual. We do not prejudge here the way of constructing questions in $\mathcal{L}$; this can be done in different ways (cf.e.g. [1], [3], [4], [8]). Yet, we assume that the language $\mathcal{L}$ satisfies certain general "erotetic" conditions. First, we assume that to each question of $\mathcal{L}$ there is assigned at least two-element set of *direct answers* to this question, which are *sentences* (d-wffs with no free variables) of $\mathcal{L}$. From the point of view of a pragmatist, direct answers are the *possible and just-sufficient* answers to the question. Yet, we assume that direct answers are defined in syntactic terms. Second, we assume that $\mathcal{L}$ contains both *finite questions* and *infinite questions*. A question is said to be finite if and only if its set of direct answers is finite; an infinite question is a question whose set of direct answers is infinite but denumerable. Third, we assume that for each sentence A of $\mathcal{L}$ there is a question of $\mathcal{L}$ (called a *simple yes-no question*) whose set of direct answers consists entirely of the sentence A and its negation $\neg A$ (called an *affirmative direct answer* and a *negative direct answer*, respectively). We also assume that for any sentences A, B of $\mathcal{L}$ there is a question of $\mathcal{L}$ (called a *conditional yes-no question*) whose set of direct answers is made up of the sentences $A\&B$ and $A\&\neg B$, exclusively. We do not prejudge what other questions occur in $\mathcal{L}$, but we assume that $\mathcal{L}$ includes some questions different from simple yes-no questions and conditional yes-no questions.

We assume the Gödel-Bernays system of set theory. The set of (all) the direct answers to a question Q will be referred to as dQ. Sometimes we shall write A instead of $\{A\}$. We say that a question Q_1 has more direct answers than a question Q just in case the power of the set of direct answers to Q_1 is greater than the power of the set of direct answers to Q. The expression "iff" abbreviates "if and only if".

(**B**) The declarative part of the language $\mathcal{L}$ is supplemented with a standard extensional semantics. An *interpretation* of $\mathcal{L}$ is an ordered pair $< \mathcal{M}, f >$, where $\mathcal{M}$ is a non-empty set (*the universe)* and f is an *interpretation function* of the usual extensional kind (defined on the set of non-logical and "non-erotetic" constants of $\mathcal{L}$). The concepts of *satisfaction* and *truth in an interpretation* are defined for d-wffs in the standard way. We do not assign truth and falsity to questions. Yet, we say that a question Q is *sound in an interpretation* $\mathfrak{M}$ of $\mathcal{L}$ iff at least one direct answer to Q is true in $\mathfrak{M}$.

We assume that the class of interpretations of $\mathcal{L}$ includes a non-empty subclass (not necessarily a proper subclass) of *normal interpreta-*

tions. When we are dealing with questions, some generalization of the concept of entailment may be useful, namely, the concept of multiple-conclusion entailment or *mc*-entailment for short (cf. [5], [6], [12]). We say that a set of *d*-wffs X of $\mathcal{L}$ *multiple-conclusion entails* a set of *d*-wffs Y of $\mathcal{L}$ (in symbols: $X \| \models Y$) iff the following condition holds:

(#) *whenever all the d-wffs in X are true in some normal interpretation of $\mathcal{L}$, then there is at least one d-wff in Y which is true in this interpretation of $\mathcal{L}$.*

A set of *d*-wffs X of $\mathcal{L}$ *entails* a *d*-wff A of $\mathcal{L}$ (in symbols: $X \models A$) iff X *mc*-entails the set $\{A\}$.

The relation $\| \models$ of *mc*-entailment is said to be *compact* if whenever $X \| \models Y$ there exist *finite* subsets X_1 and Y_1 of X and Y such that $X_1 \| \models Y_1$. It may be shown that $\models$ is compact just in case $\| \models$ is compact. We neither assume nor deny here that entailment and *mc*-entailment are compact in $\mathfrak{L}$.

3. Definition of reducibility

We are now ready to define the concept of reducibility of questions we are interested in.

DEFINITION 1. A question Q is *reducible to* a non-empty set of questions Φ iff

(i) for each direct answer A to the question Q, for each question Q^* in Φ : A *mc*-entails the set of direct answers to Q^*, and

(ii) each set made up of direct answers to the questions of Φ which contains exactly one direct answer to each question of Φ entails some direct answer to Q, and

(iii) no question in Φ has more direct answers than Q.

Roughly, the clause (i) guarantees that *if* Q is sound, then each question in the corresponding set Φ is also sound. The clause (ii) warrants that by answering the questions of Φ we can always answer the (initial) question Q, regardless of the fact which (direct) answers to the questions of Φ will appear to be acceptable. And finally, the clause (iii) requires the questions in Φ to be no more complex than Q.

For brevity, the non-emptiness clause will be omitted in the sequel.

We can easily prove

THEOREM 1. *If a question Q is reducible to a set of questions Φ, then for each normal interpretation $\mathfrak{M}$: Q is sound in $\mathfrak{M}$ iff each question of Φ is sound in $\mathfrak{M}$.*

4. Reducibility of safe questions

There are questions which are sound (i.e. have true direct answers) only in *some* normal interpretations of the language, questions which are not sound in *any* normal interpretation of the language and questions which are sound in *each* normal interpretation of the language. Following Belnap (cf. [1]), by *safe question* we mean a question which is sound in each normal interpretation of the language; a *risky question* is a question which is not safe. In other words, a question Q is safe iff the set of direct answers to Q is *mc*-entailed by the empty set; otherwise Q is risky. Let us observe that each simple yes-no question is safe, whereas some conditional yes-no questions are risky.

Theorem 1 yields

THEOREM 2. *If a question Q is reducible to a set of questions Φ, then*:
(a) *Q is safe iff each question in Φ is safe, and*
(b) *Q is risky iff some question in Φ is risky.*

Simple yes-no questions may be regarded as the simplest questions. It is interesting that each safe question can be reduced to a set of questions made up of simple yes-no questions.

THEOREM 3. *Each safe question is reducible to some set of questions made up of simple yes-no questions; each finite safe question is reducible to some finite set of questions made up of simple yes-no questions.*

PROOF. If Q is a safe question, then $\emptyset | \models dQ$. The set dQ is an at least two-element set; moreover, the set dQ is either finite or infinite but denumerable. Let $A_1, A_2 \ldots$ be a fixed enumeration of the elements of dQ. We shall define the following set of questions:

$$\Phi = \{Q : dQ = \{A_i, \neg A_i\}, \text{ where } i > 1\}$$

Since each question has at least two direct answers, Φ is non-empty; if dQ is finite, then Φ is also finite. Since each question of Φ is safe, then for each $A \in dQ$ and for each $Q_i \in \Phi$ we have $A \| \models dQ_i$. Assume that X is a fixed set made up of direct answers to the questions of Φ which contains exactly one direct answer to each question of Φ. There are two possibilities: (a) X contains some affirmative direct answer(s) to some question(s) of Φ, or (b) X consists of the negative direct answers to the questions of Φ. If the possibility (a) holds, then - since the affirmative direct answers to the questions of Φ are at the same time direct answers to Q - the set X entails some direct answer(s) to Q. If the possibility (b) holds, then - since $\emptyset \| \models dQ$ - the set X entails the direct answer A_1 to Q. □

In some cases each safe question can be reduced to some finite set of simple yes-no questions. We can prove

THEOREM 4. *If entailment is compact, then each safe question is reducible to a finite set of questions made up of simple yes-no questions.*

PROOF. If entailment is compact, then also $\| \models$ is compact; since Q is safe, it follows that there is a non-empty and finite subset Y of dQ such that $\emptyset \| \models Y$. If Y is a unit set, then there is a direct answer, say, B, to Q such that $\emptyset \models B$; hence both $B \models B$ and $\neg B \models B$. Moreover, each direct answer to Q *mc*-entails the set $\{B, \neg B\}$. Thus Q is reducible to some unit set which has a simple yes-no question as its member.

If Y is not a unit set, we proceed as in the proof of Theorem 3. □

5. Reducibility of risky questions

Theorem 2 yields that risky questions cannot be reduced to sets made up of simple yes-no questions. However, we may prove that in some cases they can be reduced to sets made up of conditional yes-no questions.

Let us introduce some supplementary concepts.

A question Q is said to be *sound relative to* a set of d-wffs X iff the set X *mc*-entails the set of direct answers to Q. (Let us stress that the above concept of relative soundness must be carefully distinguished from the concept of soundness of a question in an interpretation of the language; these are different concepts.) Following Belnap, by a *presupposition* of a question Q we mean any d-wff which is entailed by each direct answer to Q. We say that a question Q is sound relative to some of its presuppositions iff there is a presupposition C of Q such that Q is sound relative to the (unit) set $\{C\}$.

THEOREM 5. *Each risky question which is sound relative to some of its presuppositions is reducible to some set of questions made up of conditional yes-no questions.*

PROOF. Let Q be a risky question. The set dQ has at least two elements and is either finite or infinite but denumerable. Let $A_1, A_2, \ldots$ be a fixed enumeration of dQ. If Q is sound relative to some of its presuppositions, then $\mathcal{Q}$ is also sound relative to some of its presuppositions being a sentence (closed d-wff). Let B be a fixed presupposition of Q such that B is a sentence and $B| \models dQ$. We define the following set of conditional yes-no questions

$$\Phi = \{Q : dQ = \{B\&A_i, B\&\neg A_i\}, \text{ where } i > 1\}.$$

Since each question has at least two direct answers, the set Φ is non-empty. Since B is a presupposition of Q, then for each direct answer A to Q and for each question Q_i of Φ we have $A| \models dQ_i$. Let Y be an arbitrary but fixed set made up of direct answers to the questions of Φ which contains exactly one direct answer to each question of Φ. If Y contains some direct answer of the form $B\&A_i$, where $i > 1$, then Y entails some direct answer to Q. If Y consists of all the direct answers of the form $B\&\neg A_i$, where $i > 1$, then – since B *mc*-entails the set dQ - the set Y entails the direct answer A_1 to Q. Thus Q is reducible to Φ. Let us observe that if Q is a finite question, then the set Φ defined in the above manner is also finite and non-empty. □

If Q is a finite questions, then Q is sound relative to any disjunction of its direct answers and this disjunction is a presupposition of Q. Thus we can also prove

THEOREM 6. *If Q is a finite risky question, then Q is reducible to a finite set of questions made up of conditional yes-no questions.*

A question Q is said to be *normal* iff the set of presuppositions of Q *mc*-entails the set of direct answers to Q. In other words, a question Q is normal just in case Q is sound relative to the set of its presuppositions.

THEOREM 7. *If entailment is compact and Q is a risky but normal question,* *then Q is reducible to a finite set of questions made up of conditional yes-no questions.*

The proof is similar to that of Theorem 5 (if entailment is compact, then each normal risky question is sound relative to some of its presuppositions).

Note finally that Theorems 3 - 7 can be strengthened: we may prove that in each of the above cases the initial question Q is reducible to a set of questions which are implied (in the sense of [10]) by Q.

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